

ABOVE AND BELOW SUBGROUPS OF A LATTICE-ORDERED GROUP

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ABSTRACT. In an l -group G , this paper defines an l -subgroup A to be *above* an l -subgroup B (or B to be *below* A) if for every integer n , $a \in A$, and $b \in B$, $n(|a| \wedge |b|) \leq |a|$. It is shown that for every l -subgroup A , there exists an l -subgroup B maximal below A which is closed, convex, and, if the l -group G is normal-valued, unique, and that for every l -subgroup B there exists an l -subgroup A maximal above B which is *saturated*: if $0 = x \wedge y$ and $x + y \in A$, then $x \in A$.

Given l -groups A and B , it is shown that every lattice ordering of the splitting extension $G = A \times B$, which extends the lattice orders of A and B and makes A lie above B , is uniquely determined by a lattice homomorphism π from the lattice of principal convex l -subgroups of A into the cardinal summands of B . This extension is sufficiently general to encompass the cardinal sum of two l -groups, the lex extension of an l -group by an o -group, and the permutation wreath product of two l -groups.

Finally, a characterization is given of those abelian l -groups G that *split off below*: whenever G is a convex l -subgroup of an l -group H , H is then a splitting extension of G by A for any l -subgroup A maximal above G in H .

0. Introduction. Though the groups are not in general abelian, the notation is additive with the lattice operations taking priority: $a + b \wedge c$ is $a + (b \wedge c)$, $-b \wedge c$ is $(-b) \vee (-c)$, but $na \wedge b$ is $(na) \wedge b$. g^+ , g^- , and $|g|$ denote as usual $g \vee 0$, $-g \vee 0$, and $g \vee (-g)$, respectively. g is a *component* of h if $|g| \wedge |h - g| = 0$; we remind the reader that components commute and that $x + y = x \vee y$ whenever x and y are *disjoint*, that is, whenever $x \wedge y = 0$. $\mathcal{C}(G)$, $\mathcal{C}_h(G)$, $\mathcal{X}(G)$, $\mathcal{S}(G)$, and $\mathcal{P}(G)$ denote respectively the lattices of convex l -subgroups, principal convex l -subgroups, closed convex l -subgroups, cardinal summands, and polars. (The reader unfamiliar with these basic concepts can find them in [6].) $\mathcal{P}(G)$ is a complete Boolean algebra with $\mathcal{S}(G)$ as a subalgebra. We write $A \leq B$ to indicate that A is an l -subgroup of G , and $G(a)$ for $\{g \in G: |g| \leq n|a| \text{ for some integer } n\}$, the principal convex l -subgroup of G generated by a . If A is any subset of G we use $\langle A \rangle$ to designate the l -subgroup generated by A . Note that the elements of $\langle A \rangle$ have the form $\bigvee_{i=1}^n \bigwedge_{j=1}^m x_{ij}$, where the x_{ij} 's are from the subgroup generated by A . Finally, $b \ll a$ means that $n|b| \leq |a|$ for all integers n .

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A subgroup P of G is *regular* if it is maximal in $\mathcal{C}(G)$ with respect to omitting some element g , and we say that P is a *value* of g in this case. Such a subgroup is *prime* inasmuch as it contains at least one element from each disjoint pair. A set Γ of regular subgroups is *plenary* provided that $\bigcap \Gamma = 0$ and that $P \in \Gamma$ and $P \subseteq Q$ for a regular subgroup Q imply $Q \in \Gamma$. Such a set forms a *root system* in the inclusion order; that is, a partially ordered set such that for any $\gamma \in \Gamma$ the set $\{\delta \in \Gamma: \delta \geq \gamma\}$ is a chain. Most often we view Γ as an index set and use small Greek letters to indicate its elements; this abuse of notation allows us to write the corresponding regular prime as G_γ for $\gamma \in \Gamma$ and to use $G^\gamma = \bigcap \{G_\delta: \delta \in \Gamma \text{ and } G_\gamma \subset G_\delta\}$ for its cover. Note that G^γ is prime but not in general regular, and so G^γ need not correspond to a member of Γ . G is said to be *normal-valued* if each G_γ is normal in G^γ . If B is any subset of G , then $\Gamma(B)$ designates $\{\gamma \in \Gamma: G_\gamma \text{ is a value of some } b \in B\}$. An element g is *special* if $\Gamma(g)$ is a singleton, and in this case we term its value special also.

Given l -groups A and B their *cardinal sum* $A \boxplus B$ is their group direct sum with cardinal order: $a + b \geq 0$ if and only if $a \geq 0$ and $b \geq 0$. If $\{G_\lambda: \lambda \in \Lambda\}$ is a set of l -groups, then their *cardinal product* (it should be called the large sum but is not) is $\Pi G_\lambda = \{f: \Lambda \rightarrow \bigcup G_\lambda: f(\lambda) \in G_\lambda \text{ for all } \lambda \in \Lambda\}$ with componentwise group and lattice operations. The *cardinal sum* (or small sum) is $\{f \in \Pi G_\lambda: \{\lambda \in \Lambda: f(\lambda) \neq 0\} \text{ is finite}\}$. Now suppose Γ is any root system and S any subgroup of the real numbers \mathbf{R} . $V(\Gamma, S)$ designates $\{f: \Lambda \rightarrow S: \text{supp}(f) \text{ admits no infinite ascending chains}\}$, where $\text{supp}(f) = \{\lambda \in \Lambda: f(\lambda) \neq 0\}$. The group operations in $V(\Gamma, S)$ are componentwise and the order is determined by declaring $f \geq 0$ if and only if $f(\lambda) > 0$ for all maximal elements λ of $\text{supp}(f)$. The l -groups $V(\Gamma, S)$ are universal for abelian l -groups [12]. $\Sigma(\Gamma, S) = \{g \in V(\Gamma, S): \text{supp}(g) \text{ is finite}\}$.

Now let \mathcal{B} be a Boolean algebra. $1_{\mathcal{B}}$ will denote the greatest element of \mathcal{B} and $0_{\mathcal{B}}$ the least element. If $\mathcal{B} = \mathcal{P}(G)$ for an l -group G , then, $1_{\mathcal{B}} = G$ and $0_{\mathcal{B}} = 0$.

1. Above and below subgroups. In this section we characterize in various ways what it means for an l -subgroup A to lie above another l -subgroup B in an l -group G . We show that for any l -subgroup A there is an l -subgroup B maximal below A which is convex, closed, and unique in the normal valued case, and that for any l -subgroup B there is an l -subgroup A maximal above B which is closed and saturated. These results are fundamental for the rest of the paper.

PROPOSITION 1.1. *The following are equivalent for elements a and b in G .*

- (a) $|a| \wedge |b| \ll |a|$.
- (b) $n|b| \wedge |a|$ is a component of $n|b|$ for all positive integers n .
- (c) There is no prime P and no integer n for which $P < P + |a| < P + nb$.

If these conditions hold, then $G(b) = (G(b) \cap G(a)) \boxplus (G(b) \cap a^\perp)$, and we designate the projection l -homomorphisms $\nu a: G(b) \rightarrow G(a)$ and $\mu a: G(b) \rightarrow a^\perp$; that is, $x\nu a \in G(a)$, $x\mu a \in a^\perp$, and $x = x\nu a + x\mu a$ for all $x \in G(b)$. In particular, $x\nu a = x^+ \wedge a^+ + x^+ \wedge a^- - x^- \wedge a^+ - x^- \wedge a^-$ for all $x \in G(b)$.

PROOF. Suppose $0 \leq a \wedge b \ll a$ and let $b_0 = b \wedge a$ and $b_1 = b - b_0 = (b - a)^+$. We claim $b_0 \wedge b_1 = 0$, for which it is sufficient to show $b_1 \wedge a = 0$. For if P is any

prime such that $P + b_1 \wedge a > P$, then $P + b > P + a > P$, whence $P + a + b \geq P + 2a > P + a$ and $P + 2b \geq P + b + a \geq P + b > P + a$, meaning

$$P + 2a \wedge (a + b) \wedge (b + a) \wedge 2b = P + 2(a \wedge b) > P + a,$$

contradicting $a \wedge b \ll a$. The claim shows that b_0 and b_1 are complementary components of b , hence nb_0 and nb_1 are complementary components of nb for any positive integer n . But $nb_1 \in a^\perp$, hence $nb \wedge a = nb_0 \wedge a = nb_0$ since $b_0 \ll a$. This shows that (a) implies (b). Now suppose (b) holds and that P is a prime for which $P < P + |a|$. Since $n|b| = n|b| \wedge |a| + z$ for some $z \in a^\perp \subseteq P$, $P + nb \leq P + n|b| = P + n|b| \wedge |a| \leq P + |a|$, which establishes (c). Finally suppose $|a| \wedge |b| \ll |a|$ fails—say $n(|a| \wedge |b|) \not\leq |a|$ for some positive integer n . Then there must be some prime P for which $P + n(|a| \wedge |b|) > P + |a|$. Observe that

$$P < P + |a| \wedge |b| \leq P + |a| < P + n(|a| \wedge |b|) \leq P + n|b| = P + nb \vee (-n)b.$$

This proves that (c) implies (a).

Consider arbitrary a and b satisfying the conditions, and fix $x \in G(b)$. Then there is some integer n for which $|x| \leq n|b|$, and the argument that (a) implies (b) above shows that

$$k(x^+ \wedge |a|) \vee k(x^- \wedge |a|) \leq k(|x| \wedge |a|) \leq k(n|b| \wedge |a|) = kn(|b| \wedge |a|) \leq |a|$$

for all positive integers k . That is, the properties above hold when b is replaced by either x^+ or x^- . Therefore there are unique elements p and q in a^\perp such that $x^+ = x^+ \wedge |a| + p = x^+ \wedge a^+ + x^+ \wedge a^- + p$ and $x^- = x^- \wedge |a| + q = x^- \wedge a^+ + x^- \wedge a^- + q$. By substituting these expressions into $x = x^+ - x^-$ one obtains $x = x^+ \vee a + (p + q) \in (G(b) \cap G(a)) \boxplus (G(b) \cap a^\perp)$. \square

If a and b satisfy the conditions of Proposition 1.1 we shall say that a lies above b or that b lies below a . More generally, we shall say that a subset $A \subseteq G$ lies above a subset $B \subseteq G$, or that B lies below A , if each element of A lies above each element of B .

PROPOSITION 1.2. *Suppose G is a normal-valued l -group having positive elements a and b and plenary set Δ of regular primes. Then $b \ll a$ if and only if $a \notin G^\delta$ for all $\delta \in \Delta(b)$, and b lies below a if and only if $b \in G_\delta$ for all $\delta \in \Delta(a)$.*

PROOF. If $b \ll a$ and $\delta \in \Delta(b)$, then $G_\delta + nb < G_\delta + a$ for all positive integers n , hence $a \notin \{g \in G: G_\delta + |g| \leq G_\delta + nb \text{ for some } n\} = G^\delta$. Conversely, if it is not true that $b \ll a$ then there is some positive integer n and $\delta \in \Delta$ with $G_\delta + nb > G_\delta + a$. Clearly a lies in the cover of the unique value of b containing G_δ . Now suppose b is below a and consider $\delta \in \Delta(a)$. Then $a \in G^\delta$ implies $a \wedge b \in G_\delta$, so $G_\delta = G + a \wedge b = G + b$, meaning $b \in G_\delta$. If b is not below a then there is some $\delta \in \Delta(a \wedge b)$ for which $a \in G^\delta$. But then $a \notin G_\delta$ implies $\delta \in \Delta(a)$, and $b \notin G_\delta$. \square

The failure of Proposition 1.2 in $A(\mathbf{R})$, the l -group of order-preserving automorphisms of the real numbers \mathbf{R} , shows that the hypothesis of normal-valuedness cannot be omitted from Proposition 1.2, or indeed from the other results of this section in which it appears.

PROPOSITION 1.3. *For each positive element a in the normal valued l -group G , $D = \{b: |b| \ll a\}$ is the unique largest subgroup of G such that $D \leq a$, and $D \in \mathcal{X}(G)$. Furthermore, $C = a^\perp \boxplus D = \bigcap \{G_\delta: \delta \in \Delta(a)\} = \{b: b \text{ lies below } a\}$ is the unique largest subgroup of G which lies below a , and $C \in \mathcal{X}(G)$.*

PROOF. Let Δ be any plenary set of regular primes of G . Consider $0 < b_1, b_2 \in D$ and $\delta \in \Delta(b_1 + b_2)$. Since $\Delta(b_1 + b_2) \subseteq \Delta(b_1) + \Delta(b_2)$, $a \notin G^\delta$ by Proposition 1.2, hence $b_1 + b_2 \ll a$. It follows that $D \in \mathcal{C}(G)$, and is therefore clearly maximal with respect to $D \leq a$. To show D closed suppose $\bigvee E = e$, where $E = \{b \in D: b \leq e\}$. By Lemma 1.17 of [4], $ne = \bigvee \{nb: b \in E\}$ for each positive integer n . Now $\{nb: b \in E\} \subseteq D \leq a$, so $ne \leq a$, proving $a \gg e \in D \in \mathcal{X}(G)$.

Propositions 1.1 and 1.2 establish $\bigcap \{G_\delta: \delta \in \Delta(a)\} = \{b: b \text{ lies below } a\} \subseteq a^\perp \boxplus D$. But for any $\delta \in \Delta(a)$, $a \notin G_\delta$ implies $a^\perp \subseteq G_\delta$, and $D \subseteq G_\delta$ by Proposition 1.2. This shows all those sets are equal. The closure of C is a straightforward consequence of the closure of a^\perp and of D . \square

PROPOSITION 1.4. *The following are equivalent for subgroups A and B of G .*

- (a) A lies above B .
- (b) bva is a component of b for each $b \in B$ and $a \in A$.
- (c) There are no elements $a \in A$ and $b \in B$ and no prime P satisfying $P < P + a < P + b$. That is, $a^+ \wedge (b - a)^+ = 0$ for all $a \in A$ and $b \in B$.

If G is normal-valued with plenary set Δ of regular primes, then the conditions above are equivalent to the condition that $\delta_1 \not\geq \delta_2$ for all $\delta_1 \in \Delta(B)$ and $\delta_2 \in \Delta(A)$, or that $B \subseteq \bigcap \{G_\delta: \delta \in \Delta(A)\}$. If, in addition, B is a convex l -subgroup, then the conditions above are equivalent to $\Delta(A) \cap \Delta(B) = \emptyset$. Under any circumstances, if A is a convex l -subgroup, then the conditions above are equivalent to $B \subseteq A^\perp$.

PROOF. The equivalence of conditions (a), (b), and (c) results from Proposition 1.1, the second assertion is a consequence of Proposition 1.3, and the second assertion readily implies the third. This last observation is also straightforward. \square

COROLLARY 1.5. *If subgroup A lies above subgroup B of G , then $A \cap B = 0$. B lies both above and below A if and only if $B \subseteq A^\perp$.*

PROOF. Suppose B lies both above and below A , and consider $a \in A$, $b \in B$. Then $|a| \wedge |b| \ll |a|$ and $|a| \wedge |b| \ll |b|$, meaning $|a| \wedge |b| = 0$ or $B \subseteq A^\perp$. \square

Suppose C is a subgroup of G . A subgroup B which is maximal among those subgroups lying below C is said to be *maximal below C* , or simply *maximal below*, and likewise a subgroup A which is maximal among those subgroups lying above C is said to be *maximal above C* , or simply *maximal above*. C is said to be *saturated* if C contains each component of each of its elements. Observe that a saturated subgroup C is a sublattice since $c \vee 0$ is a component of each $c \in C$.

THEOREM 1.6. *Suppose subgroup A lies above subgroup B in G . Then A is contained in some subgroup maximal above B , and B is contained in some subgroup maximal below A . If G is normal-valued, then the unique subgroup maximal below A is $\bigcap \{G_\delta: \delta \in \Delta(A)\}$. In any case a maximal below subgroup is a closed convex l -subgroup and a maximal above subgroup is a saturated and closed l -subgroup.*

PROOF. The first assertion is a consequence of Zorn's Lemma, and the second assertion has already been established. Suppose now that B is maximal below A in G . B must be a sublattice, for if not then there exist elements $a \in A$ and $t \in \langle B \rangle$ and prime P such that $P < P + |a| < P + t$. But $t = \bigvee_I \bigwedge_J b_{ij}$ for some $b_{ij} \in B$ and finite I and J , hence $P + t = P + b_{ij}$ for some $i \in I$ and $j \in J$. But then $P < P + |a| < P + b_{ij}$, contradicting the assumption that B lies below A and showing B to be a sublattice. To show B convex consider an arbitrary t such that $b_1 \leq t \leq b_2$ for $b_1, b_2 \in B$. Then $|t| \leq |b_1| \vee |b_2| \in B$, hence t lies below A . Since $\{t: b_1 \leq t \leq b_2, b_i \in B\}$ is a subgroup containing B , $t \in B$. To show B closed consider $z = \bigvee S$, where $S = \{b \in B^+: b \leq z\}$, and let B' be the group generated by $B \cup \{z\}$. We claim B' lies below A , for if not then by Proposition 1.4(c) $|a| \wedge (t - |a|)^+ \neq 0$ for some $t \in B'$. Now t has the form $\sum_{i=1}^n (b_i + \varepsilon_i z)$ where $b_i \in B$ and ε_i is ± 1 . By replacing t by $\sum_{i=1}^n (b_i + y_i)$, where $y_i \in S$ if $\varepsilon_i = -1$ and $y_i = z$ if $\varepsilon_i = \pm 1$, we may assume $t = \sum_{i=1}^n (b_i + z)$. Now $t = \bigvee U$, where $U = \{\sum_{i=1}^n (b_i + s): s \in S\}$, hence $|a| \wedge (t - |a|)^+ = \bigvee \{|a| \wedge (u - |a|)^+: u \in U\}$, so there is some $u \in U$ for which $|a| \wedge (u - |a|)^+ > 0$. Since $u \in B$, this contradicts the assumption that B lies below A . We can only conclude that B is closed.

Suppose that A is maximal above B . To prove A saturated consider the element $a \in A$ with component z , and let A' be the group generated by $A \cup \{z\}$. To prove that A' lies above B it is enough to show that for each $t \in A'$ and prime P there is some $a_0 \in A$ with $P + t = P + a_0$. Since t has the form $\sum_{i=1}^n (a_i + \varepsilon_i z)$, where $a_i \in A$ and $\varepsilon_i = \pm 1$, we induct on n . Consider the case when $n = 1$ and $\varepsilon_i = 1$. Because the prime $-a_1 + P + a_1$ contains at least one of the disjoint elements $|z|$ and $|a - z|$, $P + a_1 + z$ is either $P + a_1$ or $P + a_1 + a$. The argument for $\varepsilon_i = -1$ is analogous, and the induction step yields to a simple argument using the above.

It remains to show that if A is maximal above B then A is closed. Suppose $z = \bigvee S$ for $S \subseteq A$ and let A' be the group generated by $A \cup \{z\}$. For the sake of contradiction assume that there are elements $t \in A'$ and $b \in B$ for which $t^+ \wedge (b - t)^+ > 0$. Let us say t is $\sum_{i=1}^n (a_i + \varepsilon_i z)$ for some $a_i \in A$ and $\varepsilon_i = \pm 1$. Then $\bigvee S = z$ implies $\bigvee R = t$, where $R = \{\sum_{i=1}^n (a_i + \varepsilon_i y_i): y_i \in S \text{ if } \varepsilon_i = \pm 1 \text{ and } y_i = z \text{ if } \varepsilon_i = -1\}$, and it follows that

$$t^+ \wedge (b - t)^+ = \bigvee \{r^+ \wedge (b - r)^+: r \in R\}.$$

Choose a specific $r = \sum_{i=1}^n (a_i + \varepsilon_i y_i) \in R$ such that $r^+ \wedge (b - r)^+ > 0$. Observe that $r^+ \wedge (b - r)^+ > 0$ also. Again, $\bigvee S = z$ implies $r = \bigwedge U$, where $U = \{\sum (a_i + \varepsilon_i w_i): w_i = y_i \text{ if } \varepsilon_i = 1 \text{ and } w_i \in S \text{ if } \varepsilon_i = -1\} \subseteq A$, and it follows that

$$r^+ \wedge (b - r)^+ = \bigvee \{r^+ \wedge (b - u)^+: u \in U\}.$$

Choose a specific $u = \sum_{i=1}^n (a_i + \varepsilon_i w_i) \in U$ such that $r^+ \wedge (b - u)^+ > 0$. Then $u^+ \wedge (b - u)^+ > 0$ and $u \in A$, contradicting the assumption that A lies above B . \square

If subgroup B of G is maximal below and A is maximal above B , then it need not follow that B is maximal below A . Consider $G = \mathbf{Z} \times (\mathbf{Z} \boxplus \mathbf{Z})$, the splitting extension of $\mathbf{Z} \boxplus \mathbf{Z}$ by \mathbf{Z} lexicographically ordered, where

$$-(m; 0, 0) + (0; a, b) + (m; 0, 0) = \begin{cases} (0; a, b) & \text{if } m \text{ is even,} \\ (0; b, a) & \text{if } m \text{ is odd.} \end{cases}$$

This is a simple example of a Scrimger group [15], studied in connection with l -group varieties; it is also a simple instance of the general construction of §3 and of Theorem 3.21 in particular. Let $B = 0 \bar{\times} (\mathbf{Z} \boxplus 0)$, a subgroup maximal below $\{(m; 0, n) : m \text{ even}\}$, and let $A = \mathbf{Z} \bar{\times} (0 \boxplus 0)$, another subgroup maximal above B . The point is that B is proper in $0 \bar{\times} (\mathbf{Z} \boxplus \mathbf{Z})$ which is maximal below A .

G is *projectable* if $\mathcal{P}(G) = \mathcal{S}(G)$. This concept is also termed strong projectability in the literature. A subset $Z \subseteq G^+$ is called *type \mathcal{U}* if

$$\bigcap \{((Z - z) \vee 0)^{\perp\perp} : z \in Z\} = \bigcap \{((-z + Z) \vee 0)^{\perp\perp} : z \in Z\} = 0_{\mathcal{P}}$$

[1]. For example, a pairwise disjoint subset of G^+ is type \mathcal{U} . G is said to be *sup \mathcal{U} complete* if every type \mathcal{U} subset of G^+ has a supremum in G , and G is said to be *laterally complete* if every pairwise disjoint subset of G^+ has a supremum in G^+ . The following proposition relates these concepts, and impinges on the results of §6, but is otherwise unrelated to the subject of this section.

PROPOSITION 1.7. *A projectable l -group G is sup \mathcal{U} complete if and only if it is laterally complete.*

PROOF. Any sup \mathcal{U} complete l -group is laterally complete. Now suppose G is laterally complete and projectable, and consider a type \mathcal{U} subset $Z \subseteq G^+$. Let \mathcal{U} be $\{\mathcal{R} \subseteq \mathcal{P}(G) : \mathcal{R} \text{ is pairwise disjoint, and for each } R \in \mathcal{R} \text{ there is some } z \in Z \text{ with } R \subseteq ((Z - z) \vee 0)^{\perp}\}$. Now \mathcal{U} is closed with respect to unions of chains, and so \mathcal{U} contains a maximal element \mathcal{R} . Since $\bigvee \{((Z - z) \vee 0)^{\perp} : z \in Z\} = 1_{\mathcal{P}}$ it follows that $\bigvee \mathcal{R} = 1_{\mathcal{P}}$. For each $R \in \mathcal{R}$ let a_R and z_R satisfy $z_R \in Z$, $R \subseteq ((Z - z_R) \vee 0)^{\perp}$, $a_R \in R$, and $z_R - a_R \in R^{\perp}$. Then $a = \bigvee \{a_R : R \in \mathcal{R}\}$ exists by the lateral completeness of G . We shall prove that $\bigvee Z = a$, the first step being to show $Z \leq a$. Suppose for contradiction that $a \not\leq z$ for some $z \in Z$; find $T \in \mathcal{R}$ such that $T \cap ((z - a)^+)^{\perp\perp} \neq 0_{\mathcal{P}}$, and let P be some prime subgroup satisfying $P < P + t$, where $t \in T^+$ and $t \leq (z - a)^+$. Then $P \leq P + a < P + z$. Now $a - a_T \in T^{\perp}$, for otherwise $\{a_R : R \in \mathcal{R}\} \leq -s + a < a$ for any $s \in T$ with $0 < s \leq a - a_T$, so $a - z_T = (a - a_T) - (z_T - a_T) \in T^{\perp} \subseteq P$, meaning $P + z_T < P + z$. But $T \subseteq ((Z - z_T) \vee 0)^{\perp}$ implies $(z - z_T) \vee 0 \in T^{\perp} \subseteq P$, which in turn implies $P + z \leq P + z_T$. This contradiction proves $Z \leq a$. $\bigvee Z = a$ by Lemma 5.6 of [1] since, as has just been pointed out, $T \subseteq (a - z_T)^{\perp}$ for all $T \in \mathcal{R}$, and so $\bigvee \{(a - z)^{\perp} : z \in Z\} = 1_{\mathcal{P}}$. \square

If $A \subseteq G$ we say that the *polars of \mathcal{A} and \mathcal{G} correspond* whenever the intersection map is one-one from $\mathcal{P}(G)$ onto $\mathcal{P}(A)$. This is equivalent to having each nontrivial polar of G intersect A nontrivially.

COROLLARY 1.8. *Suppose A is a maximal above l -subgroup of G . If G is projectable then so is A . If G is sup \mathcal{U} complete and polars in G and A correspond, then A is sup \mathcal{U} complete.*

PROOF. Suppose G is projectable, P is a polar of A and $Q = P^{\perp\perp}$ in G . For $a \in A$ let $a = a_1 + a_2$, where $a_1 \in Q$ and $a_2 \in Q^{\perp}$. Now a_1 lies in A by virtue of the saturation of A , and so $a_1 \in Q \cap A = P$. Next suppose G to be sup \mathcal{U} complete

and consider a type \mathcal{U} subset $Z \subseteq A^+$. Because polars correspond, Z is also of type \mathcal{U} considered as a subset of G , and so has a supremum $z \in G$. z must lie in A by the closure of A . \square

The relation of one subgroup being maximal above another has a tidy characterization in an abelian l -group. This characterization is important for §§5 and 6.

PROPOSITION 1.9. *Let G be an abelian l -group with plenary set Δ of regular subgroups, and let l -subgroup A lie above l -subgroup B of G . Then A is maximal above B if and only if for each $g \in G \setminus A$ there is some $a \in A$ and positive integer n such that $ng - a$ has a value in $\Delta(\bar{B})$.*

PROOF. If A is maximal above B , then it is maximal above \bar{B} by Theorem 1.6, and if $g \in G \setminus A$, then by Proposition 1.4 there is an element $ng - a$ of the subgroup generated by $A \cup \{g\}$ with $\Delta(ng - a) \cap \Delta(\bar{B}) \neq \emptyset$. The converse also follows from Proposition 1.4. \square

COROLLARY 1.10. *Let G be an abelian l -group with maximal above subgroup A . Then $ng \in A$ implies $g \in A$ for any $g \in G$ and nonzero integer n . Therefore A is divisible if G is.*

PROOF. A is maximal above $B = \bigcap \{G_\delta : \delta \in \Gamma(A)\}$. If $g \notin A$, then there is some $a \in A$ and positive integer k for which $kg - a$ has a value in $\Gamma(B)$. But then $n(kg - a) = k(ng) - na \in A$ has a value in $\Gamma(B)$ also, a contradiction to the fact that A lies above B . \square

COROLLARY 1.11. *Let G be a divisible abelian l -group with plenary set Δ of regular subgroups, and let l -subgroup A lie above l -subgroup B of G . Then A is maximal above B if and only if for each $g \in G \setminus A$ there is some $a \in A$ such that $g - a$ has a value in $\Delta(\bar{B})$.*

PROOF. Suppose A is maximal above B and that $g \in G \setminus A$, and find $a_1 \in A$ and positive integer n such that $ng - a_1$ has value $\delta \in \Delta(\bar{B})$. Since G is divisible there is an element $a \in G$ such that $na = a_1$, and $a \in A$ by the previous corollary. Since $n(g - a) = ng - a_1$, $g - a$ has value δ . \square

COROLLARY 1.12. *A maximal above subgroup A is convex if and only if A is a polar if and only if A is also maximal below.*

PROOF. If A is maximal above B and convex, then $A \subseteq B^\perp$ by Proposition 1.4, and since B^\perp lies above B by Corollary 1.5, $A = B^\perp$ and so A is a polar. But any polar is maximal below its complementary polar by Corollary 1.5. And any maximal below subgroup is convex by Theorem 1.6. \square

Given subgroup B of G let $\mathcal{A}(B)$ designate the collection of all subgroups of G maximal above B . Note that $\mathcal{A}(B) \neq \emptyset$ since the o -subgroup lies above B and is therefore contained in some subgroup maximal above B by Theorem 1.6.

PROPOSITION 1.13. $\bigcap \mathcal{A}(B) \subseteq B^\perp$ for any subgroup B of G . Therefore if B has only one subgroup maximal above it, then this subgroup is B^\perp . G is representable if and only if $\bigcap \mathcal{A}(B) = B^\perp$ for all subgroups B of G .

PROOF. Consider $0 < a \in \cap \mathcal{A}(B)$. By Theorem 1.6 we may assume $B \in \mathcal{C}(G)$. Then for any $0 < b \in B$, $\langle a + a \wedge b \rangle$ lies above B , for if $P < P + a + a \wedge b < P + b_1$ for some $b_1 \in B$ and prime P , then $P \leq P + a \leq P + a + a \wedge b < P + b_1$ implies $a \in P$, hence $a \wedge b \in P$, contradicting $P < P + a + a \wedge b$. Therefore there is some $A \in \mathcal{A}(B)$ such that $a + a \wedge b \in A$. But then $a \in A$ implies $a \wedge b \in A \cap B = 0$. This argument shows that $\cap \mathcal{A}(B) \subseteq B^\perp$. Now suppose B is a subgroup of the representable l -group G having plenary set Δ , and consider $A \in \mathcal{A}(B)$. Because B^\perp is normal in G , the subgroup generated by $B^\perp \cup A$ is $B^\perp + A$; consider an arbitrary $c + a \in B^\perp + A$ and $\delta \in \Delta(B)$. Since $B \not\subseteq G_\delta$ it follows that $B^\perp \subseteq G_\delta$, hence $G_\delta + (c + a) = G_\delta + a$, so $a \notin G_\delta$. This shows that $B^\perp + A$ lies above B , and by maximality that $B^\perp \subseteq A$. Thus $\cap \mathcal{A}(B) = B^\perp$. Finally, if G is not representable then it is only because it contains elements $b, g > 0$ for which $(g + b - g) \wedge b = 0$. We claim $\langle g \rangle$ lies above $\langle b \rangle$, for if $P < P + g < P + nb$ for some positive integer n , then $P + g - nb - g < P - g < P$, hence $P < P + (g + nb - g) \wedge b$, a contradiction. Therefore there is some $A \in \mathcal{A}(\langle b \rangle)$ such that $g \in A$. But $b^\perp \subseteq A$ would give $g + b - g \in A$, which leads to the contradiction $b \in A \cap \langle b \rangle = 0$. That is, $\cap \mathcal{A}(\langle b \rangle) \neq b^\perp$. \square

COROLLARY 1.14. *For an l -group G the following are equivalent.*

- (a) *G is archimedean.*
- (b) *Each maximal above subgroup is convex.*
- (c) *Each maximal above subgroup is a polar.*
- (d) *Each subgroup B has a unique subgroup maximal above it.*
- (e) *Each closed convex l -subgroup is maximal above and below.*

PROOF. If G is archimedean and $a, b > 0$, then $a \wedge b \ll a$ if and only if $a \wedge b = 0$. Thus (a) implies (b), and we have proven the equivalence of (b), (c), and (d) in the last two results. To show that (d) implies (a) suppose that G is not archimedean, say $0 < b \ll a$. Then $\langle a \rangle$ and $\langle a + b \rangle$ both lie above $\langle b \rangle$, and yet a and $a + b$ cannot both belong to the same subgroup A maximal above $\langle b \rangle$, for then $b \in \langle b \rangle \cap A = 0$, a contradiction. Finally, to prove the equivalence of (a) and (e) it is enough to recall a theorem of Bigard [5]: G is archimedean if and only if each closed convex l -subgroup is a polar. \square

The following example from [7] sheds light on some of the ideas we have been developing. It also nicely illustrates Corollary 1.14 by barely violating each of its conditions. Let $\Gamma = \{1, 2, 3, \dots, \omega, \alpha\}$, partially ordered by declaring only that $\omega > \alpha$. Let $G = \{g \in V(\Gamma, \mathbf{R}) : \lim_{n \rightarrow \infty} g(n) = g(\omega)\}$. Let $H = \{g \in G : g(\omega) = 0\}$. One readily checks that G is a nonarchimedean l -group all of whose convex l -subgroups are closed. The maximal below subgroups are precisely the polars, however, and in particular the only subgroup above H is (0) . The polar $G_1 = \{g \in G : g(n) = 0 \text{ for all } n = 1, 2, 3, \dots\}$ has numerous subgroups maximal above it, no one of which is convex. Also observe that H is the join in $\mathcal{X}(G)$ of the polars $E = \{g \in G : g(n) = 0 \text{ for all odd integers } n\}$ and $O = \{g \in G : g(n) = 0 \text{ for all even integers } n\}$. Thus the maximal below subgroups do not constitute a sublattice of $\mathcal{X}(G)$.

In some instances an l -subgroup A will lie above an l -subgroup B provided only that $A \cap B = 0$. The remainder of this section is devoted to an investigation of some of these instances.

LEMMA 1.15. *Suppose $t, g \in G^+$ with $g \wedge (t - g)^+ > 0$ and t special. Then $2t \wedge g$ is a nonzero component of g .*

PROOF. Consider a prime subgroup R omitting $g \wedge (t - g)^+$. Then $R < R + g < R + t$, hence $R \subseteq Q$, where Q is the unique value of t , and this implies $Q + g \leq Q + t < Q + 2t$. If $u = (g - (2t \wedge g)) \wedge (2t \wedge g) > 0$, then for any prime P omitting u we would have $P < P + 2t < P + g$. Again, the fact that $t \notin P$ implies $P \subseteq Q$, which in turn implies $Q + 2t \leq Q + g$, a contradiction. Therefore $u = 0$, proving $2t \wedge g$ to be a component of g . $2t \wedge g > 0$ because $R + (2t \wedge g) = R + g > R$, where R has the meaning above. \square

LEMMA 1.16. *Suppose $a, b \in G^+$ with $r = a \wedge (b - a)^+ > 0$. If t is a nonzero component of b such that $t^{\perp\perp} \subseteq r^{\perp\perp}$, then $t \wedge a$ is a nonzero component of a .*

PROOF. Let us show first that $u = (a - (a \wedge t)) \wedge (a \wedge t) = 0$. If $u > 0$, then since $u \leq t \in r^{\perp\perp}$ we may find a prime P omitting $u \wedge r$, from which follows $P < P + t < P + a$. Now since t is a component of b we have $P + t = P + b < P + a$, hence $(b - a)^+ \in P$. But this contradicts the assumption that $P < P + r \leq P + (b - a)^+$. We conclude $u = 0$, meaning $t \wedge a$ is a component of a . $0 \neq t^{\perp\perp} \subseteq r^{\perp\perp}$ implies $0 \neq t \wedge r \leq t \wedge a$. \square

PROPOSITION 1.17. *Suppose that A and B are l -subgroups of G such that B is convex and A is saturated. Suppose further that B satisfies either (a) or (b) below. Then A lies above B if and only if $A \cap B = 0$.*

(a) $b = \vee(T \wedge b)$ for all $b \in B^+$, where $T = \{t \in B^+ : t \text{ special}\}$.

(b) $\mathcal{S}(B)$ is dense in $\mathcal{P}(B)$. That is, for each $0 \neq X \in \mathcal{P}(B)$ there is some $0 \neq Y \in \mathcal{S}(B)$ such that $Y \subseteq X$.

PROOF. Suppose that B satisfies (a) and that $r = a \wedge (b - a)^+ > 0$ for some $a \in A^+$ and $b \in B^+$. Since $r = \vee\{(t \wedge (b - a))^+ \wedge a : t \in T\}$, we may assume $b \in T$. By Lemma 1.15, $2b \wedge a$ is a nonzero component of A . $2b \wedge a \in A$ by saturation and $2b \wedge a \in B$ by convexity, so $A \cap B \neq 0$. Now suppose that B satisfies (b) and that $r = a \wedge (b - a)^+ > 0$ for some $a \in A^+$ and $b \in B^+$. Let $X = r^{\perp\perp}$ and find $0 \neq Y \in \mathcal{S}(B)$ with $Y \subseteq X$. Let $b = t + (b - t)$, where $t \in Y$ and $b - t \in Y^\perp$. By Lemma 1.16, $t \wedge a$ is a nonzero component of a , hence $t \wedge a \in A \cap B \neq 0$. \square

Let \mathbf{S} designate the class of l -groups G whose every positive element is a disjoint supremum of special elements, or equivalently, those l -groups G having a plenary set of special subgroups. It is known (Theorem 3.18 of [3]) that G is normal-valued and completely distributive if and only if G is a large l -subgroup of a member of \mathbf{S} .

COROLLARY 1.18. *Suppose that A and B are l -subgroups of G such that B is convex and that A is saturated. Suppose further that either $B \in \mathbf{S}$ or B is projectable. Then A lies above B if and only if $A \cap B = 0$.*

We next show that G is normal valued if and only if G is an l -subgroup of a member of \mathbf{S} . The proof modifies that of Proposition 3.17 of [3], and requires some auxiliary ideas.

Suppose Δ is a plenary set of regular subgroups of G which is *normal*, that is, $-g + G_\delta + g \in \Delta$ whenever $G_\delta \in \Delta$ and $g \in G$. Then $\mathcal{C}(\Delta) = \{Y \subseteq G: Y \supseteq \bigcap \Delta_0 \text{ for finite } \Delta_0 \subseteq \Delta\}$ is the neighborhood filter of 0 of a topology (also designated $\mathcal{C}(\Delta)$) with respect to which all group and lattice operations are continuous [4]. The subbasic open sets have the form $G_\delta + g$ for $G_\delta \in \Delta$ and $g \in G$. We write $G \leq_\Delta H$ to mean that H is an l -group having a normal plenary set Δ' of regular subgroups such that G is an l -subgroup of H dense with respect to $\mathcal{C}(\Delta')$ and such that Δ' and Δ are in one-one correspondence by intersection. The density condition may be reformulated as follows: any finite collection of cosets of elements in Δ' which has nonempty intersection in H must have nonempty intersection in G also.

PROPOSITION 1.19. *Suppose Δ is a normal plenary set of regular subgroups of an l -group G . Then there is an l -group G^Δ which is unique up to l -isomorphism over G with respect to the following property. For any l -group H , $G \leq_\Delta H$ if and only if there is an l -monomorphism $\theta: H \rightarrow G^\Delta$ over G .*

PROOF. Let G^Δ be the completion of G with respect to the l -Cauchy structure $\mathcal{D}(\Delta) = \{\mathcal{F}: \mathcal{F} \text{ is a filter on } G \text{ such that } (\mathcal{F} - \mathcal{F}) \cap (-\mathcal{F} + \mathcal{F}) \supseteq \mathcal{C}(\Delta)\}$ induced by $\mathcal{C}(\Delta)$ [1]. For each $G_\delta \in \Delta$ let $\text{cl}(G_\delta) = \{[\mathcal{F}]: G_\delta \in \mathcal{F}\}$, a prime subgroup of G^Δ , and let $\Delta' = \{\text{cl}(G_\delta): \delta \in \Delta\}$. It can readily be verified that $G \leq_\Delta G^\Delta$. Now suppose $G \leq_\Delta H$ and consider $h \in H$. Let \mathcal{F} be the trace of the neighborhood filter (with respect to $\mathcal{C}(\Delta')$) of h on G , and let \rightarrow designate convergence in $\mathcal{C}(\Delta')$. Then $\mathcal{F} - \mathcal{F} \rightarrow h - h = 0$ and $-\mathcal{F} + \mathcal{F} \rightarrow -h + h = 0$, so $\mathcal{F} \in \mathcal{D}(\Delta)$. Define $h\theta = [\mathcal{F}]$. One easily verifies that θ is an l -monomorphism. \square

PROPOSITION 1.20. *G is normal valued if and only if $G^\Delta \in \mathbf{S}$, and in this case Δ' is the minimal plenary set of special subgroups of G^Δ .*

PROOF. Let H be G^Δ , and consider $0 < h \in H$. Then $h = [\mathcal{H}]$ for some Cauchy filter \mathcal{H} on G . Fix $\delta \in \Delta$ for which H_δ is a value of h in Δ' . Let \mathcal{F} be $\mathcal{M} \cup \mathcal{N}$, where $\mathcal{M} = \{G_\gamma + g_\gamma \in \mathcal{H}: G_\gamma \subseteq G_\delta\}$ and $\mathcal{N} = \{G_\gamma: G_\gamma \text{ unrelated to } G_\delta\}$. We claim that every finite subset of \mathcal{F} has nonempty intersection. Given finite $\mathcal{N}_1 \subseteq \mathcal{N}$ find $0 < x_\gamma \in G_\gamma \setminus G_\delta$ for each $G_\gamma \in \mathcal{N}_1$ and let $x = \bigwedge x_\gamma \in (\bigcap G_\gamma) \setminus G_\delta$. Given finite $\mathcal{M}_1 \subseteq \mathcal{M}$ choose $g \in \bigcap \mathcal{M}_1$, possible since $\mathcal{M}_1 \subseteq \mathcal{H}$. Because G has normal values there is some positive integer n such that $G_\delta + g < G_\delta + nx$. Therefore $nx \wedge g \in (\bigcap \mathcal{M}_1) \cap (\bigcap \mathcal{N}_1) \neq \emptyset$. This completes the proof of the claim that \mathcal{F} generates a proper filter on G , one which is readily seen to be in $\mathcal{D}(\Delta)$. Let $[\mathcal{F}] = f \in H$. Now for $\gamma \in \Delta$, $G_\gamma \subseteq G_\delta$ implies the existence of some $g_\gamma \in G$ for which $G_\gamma + g_\gamma \in \mathcal{H} \cap \mathcal{F}$, hence $G_\gamma \in \mathcal{H} - \mathcal{F}$, while G_γ unrelated to G_δ implies $G_\gamma \in \mathcal{F}$. Therefore $(\mathcal{H} - \mathcal{F}) \wedge \mathcal{F} \supseteq \mathcal{C}(\Delta)$, meaning that $(h - f) \wedge f = 0$ so that f is a component of h , and $h \in H_\gamma$ for all $\gamma \in \Delta$ such that G_γ is unrelated to G_δ , while $h - f \in H_\gamma$ for all $\gamma \in \Delta$ such that $G_\gamma \subseteq G_\delta$. Therefore h is special with sole value H_δ , which must therefore be closed. Clearly h must be the supremum of its special components. \square

To be sure, not all members of \mathbf{S} have the form G^Δ for some l -group G . It is not difficult to show, for example, that G^Δ must be $\sup \mathcal{U}$ complete and therefore laterally complete. But G^Δ need not be Dedekind MacNeille complete; in particular, if G is the l -group of rational numbers, then $G^\Delta = G$. Is it possible to characterize l -groups of the form G^Δ ?

In the next result, A^\sim designates the convexification and $[A]$ designates $\bigcap \{C : A \leq C \leq G \text{ and } C \text{ saturated}\}$ for an l -subgroup A of G .

THEOREM 1.21. *Suppose A and B are l -subgroups of the normal-valued l -group G . Then A lies above B if and only if $[A] \cap B^\sim = 0$ in G^Δ . In particular, if $G \in \mathbf{S}$, then a saturated subgroup A lies above $B \in \mathcal{C}(G)$ if and only if $A \cap B = 0$.*

PROOF. Suppose $G \in \mathbf{S}$ and that Δ is the minimal plenary set of special subgroups of G . Suppose that the saturated subgroup A fails to lie above the convex l -subgroup B . By Proposition 1.4 there is some $a \in A^+$, $b \in B^+$, and $\delta \in \Delta$ such that $\delta \in \Delta(a) \cap \Delta(b)$. Because $G \in \mathbf{S}$ we may replace a by its special component having value δ , which also lies in A because A is saturated. Because G^δ/G_δ is archimedean, $a \leq nb$ for some positive integer n . That is, $a \in A \cap B \neq \emptyset$. \square

2. Maximal below subgroups. In this section we present conditions sufficient for an l -group B to be maximal below in G . Given another l -subgroup A the notation $B \vee A^\perp$ refers to $\{g \in G : |g| \leq n(b \vee x) \text{ for some } 0 < b \in B, 0 < x \in A^\perp, \text{ and positive integer } n\}$, the supremum of B and A^\perp in $\mathcal{C}(G)$.

PROPOSITION 2.1. *For l -subgroups A and B with B convex, (a) implies (b), and (b) and (c) are equivalent. If $G = A^{\perp\perp} \boxplus A^\perp$ then (b) implies (a).*

- (a) $B \vee A^\perp$ is maximal below A in G .
- (b) $B \cap A^{\perp\perp}$ is maximal below A in $A^{\perp\perp}$.
- (c) $B \cap A^{\perp\perp}$ is maximal below $A \boxplus A^\perp$ in G .

PROOF. Assuming (a) holds, $B \cap A^{\perp\perp}$ surely lies below A in $A^{\perp\perp}$. If $B \cap A^{\perp\perp} \subsetneq C \in \mathcal{C}(A^{\perp\perp})$, then $C \vee A^\perp \supsetneq B \vee A^\perp$ by virtue of the distributivity of $\mathcal{C}(G)$; hence there are elements $a \in A^+$, $c \in C^+$, $0 < x \in A^\perp$, a prime subgroup P , and a positive integer n such that $P < P + a < P + n(c \vee x) = P + nc \vee nx = P + nc$. ($n(c \vee x) = n(c + x) = nc + nx = nc \vee nx$ because $c \wedge x = 0$, and $nx \in P$ because $a \wedge x = 0$ and $a \notin P$.) This proves that (b) holds. If (b) holds and $B \cap A^{\perp\perp} \subsetneq C \in \mathcal{C}(G)$ such that C lies below $A \boxplus A^\perp$, then $C \subseteq A^{\perp\perp}$ by the last assertion of Proposition 1.4, and so $C = B \cap A^{\perp\perp}$, meaning (c) holds, and (c) clearly implies (b). Finally, suppose that $G = A^{\perp\perp} \boxplus A^\perp$ and that (b) holds. Given $0 < a \in A$, $0 < b \in B$, and $0 < x \in A^\perp$ we may write $b = bva + b\mu a$, where $bva \ll a$. Therefore

$$a \wedge (n(b \vee x)) \leq a \wedge (n(b + x)) = a \wedge (nbva + n(b\mu a + x)) = nbva \ll a,$$

proving that $B \vee A^\perp$ lies below A . If $B \vee A^\perp \subsetneq C \in \mathcal{C}(G)$ such that C lies below A , then

$$C = C \cap G = (C \cap A^{\perp\perp}) \boxplus (C \cap A^\perp) = (B \cap A^{\perp\perp}) \boxplus A^\perp = B \vee A^\perp,$$

where the penultimate equality holds by (b). \square

The following example shows that (b) does not generally imply (a) in Proposition 2.1. Let $\Gamma = \{0, 1, 2, \dots\}$ be ordered by declaring $0 > 2n + 1$ for $n = 1, 2, 3, \dots$. Let $G = \{g \in V(\Gamma, \mathbb{Z}): \{n \in \Gamma: (n)g \neq 0\} \text{ is finite for some } k \in \mathbb{Z}\}$, let A be $\langle a \rangle$ where $(0)a = 1$ and $(n)a = 0$ for $n > 0$, and let $B = \{g \in G: (2n)g = 0 \text{ for } n = 0, 1, 2, \dots\}$. Then B is maximal below A in $A^{\perp\perp} = A$, yet $B \vee A^{\perp} = \{g \in G: 0 \notin \{n \in \Gamma: (n)g \neq 0\} \text{ finite}\}$ is properly contained in $B_1 = \{g \in G: (0)g = 0\}$, and B_1 is (maximal) below A in G .

COROLLARY 2.2. *Suppose A and B are l -subgroups of G . Then B is maximal below A in $A^{\perp\perp}$ if and only if B is maximal below $A \boxplus A^{\perp}$.*

We refer to (A, B) as a *maximal above and below pair* if A is an l -subgroup of G maximal above B and B is an l -subgroup of G maximal below A .

PROPOSITION 2.3. *Suppose (A, B) is a maximal above and below pair in G . If B is prime, then A must be totally ordered. In a normal-valued l -group G , B is prime if and only if A is a totally ordered set of special elements. In particular, if a prime subgroup of a normal-valued l -group is maximal below, then it is an intersection of special subgroups. Therefore a regular subgroup of a normal-valued l -group is maximal below if and only if it is special.*

PROOF. If a_1 and a_2 are disjoint elements of A , then one must lie in B , hence one is 0. That is, A is totally ordered. If G is normal valued, then $B = \bigcap \{G_{\delta}: \delta \in \Delta(A)\}$, hence $\Delta(A)$ must be totally ordered and all elements of A must be special. \square

To see that the normal-valued hypothesis cannot be omitted from the preceding result take G to be $\{g \in A(\mathbb{R}): g + p = p + g\}$, where $(r)p = r + 1$ for all $r \in \mathbb{R}$. Let $A = \langle a \rangle$, where $(r)a = r + 2$, and let B be $\{g \in G: (0)g = 0\}$.

COROLLARY 2.4. *A normal-valued l -group is finite-valued if and only if every regular subgroup is maximal below.*

As in §1, let \mathbf{S} designate the class of all l -groups G whose every positive element is a disjoint supremum of special elements.

COROLLARY 2.5. *If G is normal-valued, completely distributive, and if every closed convex l -subgroup of G is maximal below, then $G \in \mathbf{S}$.*

PROOF. Let Δ be the minimal plenary set of closed regular subgroups of G . Since each G_{δ} is maximal below, it is also special. \square

PROPOSITION 2.6. *A subgroup B of an abelian l -group G is prime and maximal below if and only if B is an intersection of a chain of special subgroups.*

PROOF. Suppose $B = \bigcap \{G_{\lambda}: \lambda \in \Lambda\}$ for some chain Λ of special subgroups. For each $\lambda \in \Lambda$ choose special $a_{\lambda} \in G^{\lambda} \setminus G_{\lambda}$, and let A be $\langle a_{\lambda}: \lambda \in \Lambda \rangle$. Then $B = \bigcap \{G_{\lambda}: \lambda \in \Delta(A)\}$ is maximal below A . \square

COROLLARY 2.7. *An abelian l -group is finite-valued if and only if each prime subgroup is maximal below.*

COROLLARY 2.8. *If $0 < a$ is special with value G_γ , then $G_\gamma \cap a^{\perp\perp}$ is maximal below $\langle a \rangle \boxplus a^\perp$ and also maximal below $\langle a \rangle$ in $a^{\perp\perp}$.*

PROOF. Take A to be $\langle a \rangle$ and B to be G_γ in Proposition 2.1. \square

The convex l -subgroup M of the following lemma has been well studied. A proof can be found in [11], for example, of the final assertion of the following lemma.

LEMMA 2.9. *Suppose P is a prime subgroup of G and let $M = \{g \in G: |g| \wedge q = 0 \text{ for some } q \in G \setminus P\}$. Then $M \in \mathcal{C}(G)$, and any convex l -subgroup C which contains M must be related to P . Thus $M = \bigcap \{Q \in \mathcal{C}(G): Q \subseteq P \text{ and } Q \text{ prime}\}$.*

PROOF. If $0 < g_1, g_2 \in M$ with $g_i \wedge q_i = 0$ for $q_1, q_2 \in G \setminus P$, then $(g_1 + g_2) \wedge (q_1 \wedge q_2) = 0$, and $q_1 \wedge q_2 \in G \setminus P$ by the primeness of P . Thus $M \in \mathcal{C}(G)$. Now suppose $M \subseteq C \not\subseteq P$ for some $C \in \mathcal{C}(G)$; say $c \in C \setminus P$. Consider $p \in P^+$. Then $c \wedge p \in P \cap C$, hence $c' = c(c \wedge p)^{-1} \notin P$. Because $c' \wedge p' = 0$, where $p' = p(c \wedge p)^{-1}$, we conclude that $p' \in M \subseteq C$, whence $p \in C$. That is, $P \subseteq C$. Finally, let $\mathcal{M} = \{Q \in \mathcal{C}(G): Q \subseteq P, Q \text{ prime}\}$. Given $p \in P \setminus M$ there is some prime subgroup Q such that $M \subseteq Q$ and $p \notin Q$, and the preceding argument shows $Q \in \mathcal{M}$. Therefore $\bigcap \mathcal{M} \subseteq M$. On the other hand, for any $Q \in \mathcal{M}$, $g \in G$, and $q \in G \setminus P$ such that $g \wedge q = 0$ we must have $g \in Q$ by primeness. That is, $M \subseteq \bigcap \mathcal{M}$. \square

PROPOSITION 2.10. *If G is abelian and $0 < a \in G$ is special with value G_γ , then each $N \in \mathcal{C}(G)$ between $G_\gamma \cap a^{\perp\perp}$ and $a^{\perp\perp}$ is maximal below.*

PROOF. If $N = a^{\perp\perp}$, then N is maximal below a^\perp by Proposition 1.4, and if $N = G_\gamma \cap a^{\perp\perp}$, then N is maximal below $\langle a \rangle \boxplus a^{\perp\perp}$ by Corollary 2.8, so assume $G_\gamma \cap a^{\perp\perp} < N < a^{\perp\perp}$. Observe that any $h \in a^{\perp\perp} \setminus G_\gamma$ must be special. (If h had a value G_δ unrelated to G_γ , then by Lemma 2.9 there would be disjoint elements $x \in G_\delta \setminus G_\gamma$ and $y \in G_\gamma \setminus G_\delta$, and by replacing x by some multiple we could assume $x > g$. Then $y \in g^\perp \setminus h^\perp$, contrary to assumption.) Let $\Gamma = \{G_\gamma: G_\gamma \text{ is the value of some } 0 < h \in a^{\perp\perp} \setminus N\}$ and for each choose $h_\gamma \in a^{\perp\perp} \setminus N$ with value G_γ . Let $A = \langle h_\gamma: \gamma \in \Gamma \rangle$. Since N is maximal below A in $a^{\perp\perp}$ by construction, N is maximal below $A \boxplus a^\perp$ by Proposition 2.1. \square

PROPOSITION 2.11. *Suppose $\{G_\lambda: \lambda \in \Lambda\}$ is a set of special subgroups. If G is abelian, or if each element of Λ exceeds an element minimal in Λ , then $B = \bigcap \{G_\gamma: \lambda \in \Lambda\}$ is maximal below.*

PROOF. For each $\lambda \in \Lambda$ choose a_λ to be a positive special element in $G^\lambda \setminus G_\lambda$. If G is abelian, then the subgroup A generated by the a_λ 's is the direct sum of the cyclic subgroups $\langle a_\lambda \rangle$, hence $\Gamma(A) = \Lambda$, where Γ here represents the plenary set of all regular subgroups of G . Therefore B is maximal below A by Theorem 1.6. If each element of Λ exceeds an element in $\Lambda_0 = \{\lambda \in \Lambda: \lambda \text{ is minimal in } \Lambda\}$, then let $A = \langle a_\lambda: \lambda \in \Lambda_0 \rangle$. Observe that since $a_\lambda \wedge a_\delta = 0$ for $\lambda \neq \delta$ in Λ_0 , A is the cardinal sum of the cyclic subgroups $\langle a_\lambda \rangle$ for $\lambda \in \Lambda_0$. If $0 < a \in A$ and $0 < b \in \bigcap \{G_\lambda: \lambda \in \Lambda_0\}$ satisfy $P < P + a < P + b$, then the value of a containing P must lie in Λ_0 and therefore contains b , a contradiction. This shows that $\bigcap G_\lambda$ lies below

A. If $G_\lambda \subset B \in \mathcal{C}(G)$, then there is some $\lambda \in \Lambda_0$ such that $B \not\subseteq G_\lambda$. Because G_λ is normal in G^λ (Corollary 4.3.5 and Proposition 4.4.4 of [6]), $G_\lambda + a_\lambda < G_\lambda + b$ for some $b \in B$, hence $a_\lambda \in B$, a contradiction. \square

COROLLARY 2.12. *If $\Gamma(G)$ is finite, then every convex l -subgroup is maximal below.*

LEMMA 2.13. *Suppose B is a convex l -subgroup of an abelian l -group G . If there is a plenary set Δ of regular primes such that for each $\delta \in \Delta \setminus \Delta(B)$ there is some $0 < g(\delta) \in G$ with the property that $\{\gamma \in \Delta(g(\delta)) : \gamma \text{ is related to an element of } \Delta(B)\} \subseteq \{\delta\} \subseteq \Delta(g(\delta))$, then $B' = \{g \in G : \Delta(g) \subseteq \Delta(B)\}$ is maximal below.*

PROOF. An element of the group generated by $\{g(\delta) : \delta \in \Delta \setminus \Delta(B)\}$ has the form $x = \sum_{i=1}^k n_i g(\delta_i)$, where $\delta_i \in \Delta \setminus \Delta(B)$ and $n_i \in \mathbf{Z}$. Suppose x has value γ related to an element $\lambda \in \Delta(B)$. By relabelling, assume $g(\delta_i) \notin G_\gamma$ for $1 \leq i \leq m$ and $g(\delta_i) \in G_\gamma$ for $m+1 \leq i \leq k$, so that $G_\gamma + x = G_\gamma + \sum_{i=1}^m n_i g(\delta_i)$. For each i , $1 \leq i \leq m$, $g(\delta_i)$ has a value ρ_i containing γ , and since ρ_i is also related to λ , $\rho_i = \delta_i$ in each case. Therefore γ is the greatest element in the chain $\{\delta_i : 1 \leq i \leq m\}$, and in particular $\gamma \in \Delta \setminus \Delta(B)$. Since $\Delta(A) = \Delta \setminus \Delta(B)$ it follows that B' is maximal below A . \square

PROPOSITION 2.14. *If g is a finite-valued element of an abelian l -group G , then $G(g)$ is maximal below.*

PROOF. Write g as $g_1 + g_2 + \cdots + g_n$, where the g_i 's are disjoint and special with respective values $\gamma_1, \gamma_2, \dots, \gamma_n$. Given $\lambda \in \Gamma \setminus \Gamma(g)$ let $I = \{i : \lambda \not\asymp \gamma_i\}$, and for each $i \in I$ choose $0 < x_i \in (G^\lambda \setminus G_\lambda) \cap G_{\gamma_i}$. Let $g(\lambda) = \bigwedge \{(x_i - |g_i|)^+ : i \in I\}$ if $I \neq \emptyset$, otherwise let $g(\lambda)$ be any positive element of $G^\lambda \setminus G_\lambda$. Now take $B = G(g)$ in Lemma 2.13, and note that $B' = B$. \square

The descending chain condition (every descending chain is finite) is abbreviated DCC.

PROPOSITION 2.15. *Suppose $G \in \mathbf{S}$, and suppose G is either abelian or that the minimal plenary set Δ satisfies the DCC. Then every closed convex l -subgroup is maximal below.*

PROOF. For $B \in \mathcal{X}(G)$ we claim $B = \bigcap \{G_\delta : \delta \in \Delta, B \subseteq G_\delta\}$. For if $0 < x \notin B$ then $x = \vee T$ for some disjoint set T of special elements. Since B is closed it cannot contain T , say $0 < t \in T \setminus B$ with value $\delta \in \Delta$. Then $B \subseteq G_\delta$, since otherwise $t \in B$, and this proves the claim. To prove the proposition take $\Gamma = \{\delta \in \Delta : B \subseteq G_\delta\}$ in Proposition 2.11. \square

COROLLARY 2.16. *If G is finite-valued and its plenary set of regular subgroups enjoys the DCC, then every convex l -subgroup of G is maximal below.*

The example following Theorem 2.19 shows that the DCC cannot be omitted from the hypotheses of the previous corollary. In the following proposition, $\bar{B} = \bigcap \{C \in \mathcal{X}(G) : B \subseteq C\}$ denotes the *closure* of B . It is known that $\bar{B} = \{g \in G : |g| = \vee \{b \in B^+ : b \leq |g|\}\}$.

LEMMA 2.17. *Suppose G is normal-valued and completely distributive with minimal plenary set Δ . Then $\bar{B} = \bigcap \{G_\delta : B \subseteq G_\delta \text{ and } \delta \in \Delta\} = \{g \in G : \Delta(g) \subseteq \Delta(B)\}$.*

PROOF. The first equality follows from the fact that any closed convex l -subgroup of a normal-valued completely distributive l -group is an intersection of closed prime subgroups [14]. The second equality can be routinely verified. \square

PROPOSITION 2.18. *Suppose G is an abelian l -group in \mathbf{S} . If A is a subgroup maximal above $B \in \mathcal{C}(G)$, then \bar{B} is maximal below A .*

PROOF. \bar{B} lies below A by Theorem 1.6. Suppose $\bar{B} \subsetneq H \in \mathcal{C}(G)$ and that H lies below A . Fix $0 < h \in H \setminus \bar{B}$, and write $h = \vee T$ for some disjoint set T of special elements. Now $T \not\subseteq \bar{B}$, for otherwise the closure of \bar{B} would imply $h \in \bar{B}$. Fix $t \in T \setminus \bar{B}$ with value γ . We claim that the subgroup A' generated by $A \cup \{t\}$ lies above B . Consider a value δ of a typical element $a + nt \in A'$, where $a \in A$ and n is an integer. If $t \in G_\delta$, then δ is a value of a and so $G_\delta \supseteq B$. If $t \notin G_\delta$, then $\delta \leq \gamma$, and by Proposition 1.2 $\gamma < \alpha$ for some value α of a . Therefore $\delta = \alpha$, and again $G_\delta \supseteq B$. This proves the claim. But then the maximality of A forces $t \in A$, contradicting the assumption that H lies below A . We can only conclude that $\bar{B} = H$. \square

The following theorem sums up the preceding results.

THEOREM 2.19. *Suppose G is normal-valued and completely distributive with minimal plenary set Δ . If G is either abelian or Δ has the DCC, then each closed convex l -subgroup of G is maximal below if and only if G is in \mathbf{S} . If G is abelian and in \mathbf{S} and if A is a subgroup maximal above $B \in \mathcal{C}(G)$, then $\bar{B} = \{g \in G : \Delta(g) \subseteq \Delta(B)\} = \bigcap \{G_\delta : \delta \in \Delta \text{ and } B \subseteq G_\delta\}$ is maximal below A .*

We close this section with an example which shows that the hypothesis that G be either abelian or have a root system with the DCC cannot be dropped from the last several results. This example also shows that a normal convex l -subgroup of a totally ordered group need not be maximal below. Let G be the free group on a countably infinite set of generators and let $B = [G, G]$ be its commutator subgroup. Then G/B is group isomorphic to $\sum_{i=1}^{\infty} \mathbb{Z}_i$ and G can be totally ordered so that B is convex and G/B is o -isomorphic to $\Sigma(\omega^*, \mathbb{Z})$, where ω^* is the set of integers under reverse order. The point here is that G/B has no smallest nontrivial convex l -subgroup.

Now suppose B is maximal below A and pick $u, v \in A$. Then $u + v - u - v \in A \cap B = 0$, meaning A is abelian. Since A is also free, $A = \langle a \rangle$. If G_a is the value of a in G we get $B \subsetneq G_a$ below A . Thus B is not maximal below. Observe that this example shows that the set of maximal below subgroups of G is not closed under intersection.

The construction above can be modified to produce a totally ordered group H whose every convex l -subgroup is normal, but whose only maximal below subgroups are the regular ones. Let H be the free group on countably many generators as before, and let $H = H^{(0)} \geq H^{(1)} \geq \dots$ be its descending central series. Then H can be totally ordered so that each $H^{(n)}$ is convex and so that each quotient $H^{(n)}/H^{(n+1)}$

is order isomorphic to $\Sigma(\omega^*, \mathbf{Z})$. An argument similar to the one above shows that H has the stated properties.

3. The construction of the splitting extension. Suppose l -groups A and B are given along with a group homomorphism σ from A into the group of l -automorphisms of B . Can the resulting group splitting extension of B by A , denoted $A \times B$, be endowed with a lattice order extending those on A and B and with respect to which A lies above B ? We prove that the answer to this question is affirmative, and that every such order is uniquely determined by a lattice homomorphism $\pi: \mathcal{C}_\ell(A) \rightarrow \mathcal{S}(B)$ such that $\pi(0) = 0$. The construction of $G = A \times B$ from A , B , σ , and π is shown to be sufficiently general to encompass cardinal product, lex extension, and l -permutation wreath product.

LEMMA 3.1. *Suppose a lies above b in G . Then $(a + b)^+ = a^+ + a^+ \wedge b^+ - a^+ \wedge b^- + (b\mu a)^+$ and $(a + b)^- = a^- \wedge b^- - a^- \wedge b^+ + a^- + (b\mu a)^-$. Thus $a + b \geq 0$ if and only if $a \geq 0$ and $b\mu a \geq 0$.*

PROOF. By substituting the expression for $b\nu a + b\mu a$ in Proposition 1.1 for b in $a + b$ and grouping, one obtains $a + b = c - d + b\mu a$, where $c = a^+ + a^+ \wedge b^+ - a^+ \wedge b^-$ and $d = a^- \wedge b^- - a^- \wedge b^+ + a^+$. Now $a^+ = 0$ implies $c = 0$ and $a^+ > 0$ implies $c = a^+ + a^+ \wedge b^+ - a^+ \wedge b^- \geq a^+ - a^+ \wedge b^- > 0$ because b^- lies below a^+ . Likewise $a^- = 0$ implies $d = 0$ and $a^- > 0$ implies $d > 0$. Since c and d are disjoint, we get $(a + b)^+ = c + (b\mu a)^+$ and $(a + b)^- = d + (b\mu a)^-$. Clearly $(a + b)^- = 0$ if and only if $a^- = (b\mu a)^- = 0$. \square

THEOREM 3.2. *Suppose A and B are l -subgroups of G such that A is above B and let H be the subgroup generated by A and B . If B is normal in H , then H is a splitting extension of B by A , and the following are equivalent.*

- (a) H is an l -subgroup of G .
- (b) $a \wedge b \in B$ for all $a \in A^+$ and $b \in B^+$.
- (c) $B = B \cap H(a) \boxplus B \cap a^\perp$ for all $a \in A$.

Furthermore, B is maximal below A and A is maximal above B in H . In particular, if G is normal-valued and if B is maximal below A in G , then B is normal in H and the conditions above hold.

PROOF. Suppose B is normal in H . Then H is a splitting extension of B by A because $B \cap A = 0$. Now assume (a) and consider $a \in A^+$ and $b \in B^+$. Then $a \wedge b = a_1 + b_1$ for some $a_1 \in A$ and $b_1 \in B$. We claim that $a_1 = 0$, for if not there is some prime P omitting a_1 . P also omits $a_1 + b_1$, for otherwise we would have $P < P + a_1 = P - b_1 < P - 2b_1$ or perhaps $P < P - a$, meaning $Q < Q - a_1 = Q + b_1 < Q + 2b_1$ where $Q = -a_1 + P + a_1$, in either case a violation of Proposition 1.1(c). So $P < P + a_1 + b_1 = P + a \wedge b = P + b$, hence $P < P + a_1 = P + (b - b_1) < P + 2(b - b_1)$, again contradicting Proposition 1.1(c). This proves that $a \wedge b = b_1 \in B$. If (b) holds then $b\nu a \in B$ by Proposition 1.1 and so (c) holds. If $a \in A^+$ and $b \in B^+$ and (c) holds, then $b = b_1 + b_2$ where $b_1 \in B \cap H(a)$ and $b_2 \in B \cap a^\perp$, but by Proposition 1.1, $b_1 = b\nu a = b \wedge a \in B$, hence (b) holds.

And (b) implies (a) by Lemma 3.1. Finally, observe that A is maximal above B in H in the very strong sense that any subgroup of H properly containing A has nontrivial intersection with B , and B is likewise maximal below A in H .

If B is maximal below A in the normal-valued l -group G and if $a \in A^+$ and $b \in B^+$, then $b \wedge (a + a_1 - a) \ll a + a_1 - a$ implies $(-a + b + a) \wedge a_1 \ll a_1$. This shows that B is normal in H , and the convexity of B in G implies that conditions (a), (b), and (c) hold in this case. \square

Suppose A , B , and H satisfy the conditions of Theorem 3.2. With each $a \in A$ associate the l -automorphism $\sigma(a)$ of B defined by $b\sigma(a) = -a + b + a$. Then the addition in $H = A + B$ is given by

$$(a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1\sigma(a_2) + b_2).$$

With each $a \in A$ also associate $[a] = B \cap H(a) = \{b \in B : |b| \ll |a|\} \in \mathcal{S}(B)$. Denote the complementary summand $[a]' = B \cap a^\perp$. Let $\pi: \mathcal{C}\mathcal{H}(A) \rightarrow \mathcal{S}(B)$ be defined by $\pi(A(a)) = [a]$. Note that π is well defined since $A(a_1) = A(a_2)$ implies $H(a_1) = H(a_2)$. Associated with the summand $[a]$ are the projection l -epimorphisms $\nu_a: B \rightarrow [a]$ and $\mu_a: B \rightarrow [a]'$ satisfying $b = b\nu_a + b\mu_a$ for all $b \in B$. The following proposition, whose proof is straightforward, sets down the relevant attributes of σ and π .

PROPOSITION 3.3. *Suppose A and B are l -subgroups of G such that B lies below A , B is normal in the group H generated by $A \cup B$, and H is an l -subgroup of G . Then σ is a group homomorphism from A into the group of l -automorphisms of B , π is a lattice homomorphism from $\mathcal{C}\mathcal{H}(A)$ into $\mathcal{S}(B)$, and the following properties hold.*

- (a) $\pi(0) = 0$.
- (b) $\sigma(a)$ is the identity on $[a]'$ for all $a \in A$.
- (c) $[a_1]\sigma(a_2) = [-a_2 + a_1 + a_2]$ for all $a_1, a_2 \in A$.

The following theorem, which is a converse of Proposition 3.3, is one of the central results of this paper. It asserts that the properties of σ and π pointed out above are the important ones, for they are sufficient to construct H from A , B , σ , and π .

THEOREM 3.4. *Suppose A and B are l -groups, σ is a group homomorphism from A into the group of l -automorphisms of B , and π is a lattice homomorphism from $\mathcal{C}\mathcal{H}(A)$ into $\mathcal{S}(B)$, where $\pi[A(a)]$ is denoted by $[a]$. Suppose further that properties (a), (b), and (c) of Proposition 3.3 hold. Let $G = A \times B$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1\sigma(a_2) + b_2)$, and define $(a, b) \geq 0$ if and only if $a \geq 0$ and $b\mu_a \geq 0$. Then G is an l -group in which A and B are embedded as l -subgroups such that A is maximal above B and B is maximal below A . Moreover $b\sigma(a) = -a + b + a$ and $\pi[A(a)] = [a] = G(a) \cap B$ for all $a \in A$ and $b \in B$.*

References to A and B as subgroups of G in Theorem 3.4 presume the usual identification of each $a \in A$ with $(a, 0) \in G$ and each $b \in B$ with $(0, b) \in G$. We shall prove this theorem by a sequence of lemmas, each of which assumes the notation and terminology of Theorem 3.4. The first lemma in the sequence proves

the antisymmetry of the order; it makes use of the fact that $[0]' = B$ because $[0] = \pi(0) = 0$, hence $b\nu 0 = 0$ and $b\mu 0 = b$ for all $b \in B$. Note also that $-(a, b) = (-a, -b\sigma(-a))$ in G .

LEMMA 3.5. *If $(a, b) \geq 0$ and $-(a, b) \geq 0$, then $a = 0$ and $b = 0$.*

PROOF. $(a, b) \geq 0$ implies $a \geq 0$, and $-(a, b) = (-a, -b\sigma(-a)) \geq 0$ implies $a \leq 0$, so $a = 0$. And $(a, b) \geq 0$ also implies $b\mu a = b\mu 0 = b = 0$. \square

The next lemma proves the closure of the positive cone under addition. It makes use of the fact that for $a_1, a_2 \in A$ and $b \in B$, $b\nu a_1\sigma(a_2)$ is a component of $b\sigma(a_2)$ because $b\nu a_1$ is a component of b . Since $[a_1]\sigma(a_2) = [-a_2 + a_1 + a_2]$, we see $b\nu a_1\sigma(a_2) = b\sigma(a_2)\nu(-a_2 + a_1 + a_2)$ and $b\mu a_1\sigma(a_2) = b\sigma(a_2)\mu(-a_2 + a_1 + a_2)$. In addition if $A(a_1) \subseteq A(a_2)$, then $[a_1] = \pi(A(a_1)) \subseteq \pi(A(a_2)) = [a_2]$. So under these circumstances $b\nu a_2\nu a_1 = b\nu a_1\nu a_2 = b\nu a_1$ and $b\mu a_2\mu a_1 = b\mu a_1\mu a_2 = b\mu a_2$, hence $b\nu a_2 \geq 0$ implies $b\nu a_1 = 0$ and $b\mu a_1 = 0$ implies $b\mu a_2 = 0$.

LEMMA 3.6. *If $(a_1, b_1), (a_2, b_2) \geq 0$, then $(a_1, b_1) + (a_2, b_2) \geq 0$.*

PROOF. Clearly $a_1 + a_2 \geq 0$ because $a_1, a_2 \geq 0$, and

$$\begin{aligned} (b_1\sigma(a_2) + b_2)\mu(a_1 + a_2) &= b_1\sigma(a_2)\mu(a_1 + a_2) + b_2\mu(a_1 + a_2) \\ &= b_1\mu(a_2 + a_1)\sigma(a_2) + b_2\mu(a_1 + a_2) \\ &\geq 0 \end{aligned}$$

because $b_1\mu a_1, b_2\mu a_2 \geq 0$ imply $b_1\mu(a_1 + a_2)$ and $b_2\mu(a_1 + a_2) \geq 0$. \square

The next lemma establishes the normality of the positive cone.

LEMMA 3.7. *If $(a_2, b_2) \geq 0$, then $-(a_1, b_1) + (a_2, b_2) + (a_1, b_1) \geq 0$ for any $(a_1, b_1) \in G$.*

PROOF.

$$-(a_1, b_1) + (a_2, b_2) + (a_1, b_1) = (a, -b_1\sigma(a) + b_2\sigma(a_1) + b_1),$$

where $a = -a_1 + a_2 + a_1 \geq 0$, and

$$\begin{aligned} -b_1\sigma(a)\mu a + b_2\sigma(a_1)\mu a + b_1\mu a &= -b_1\mu a\sigma(a) + b_2\mu a_2 + b_1\mu a \\ &= -b_1\mu a + b_2\mu a_2 + b_1\mu a \geq 0, \end{aligned}$$

where the last equality holds because $\sigma(a)$ is the identity on $[a]'$. \square

Thus far we have shown G to be a partially ordered group. We shall show outright that G is a lattice in Lemma 3.9, which is made easier by the following result.

LEMMA 3.8. *$(a_1, b_1) \geq (a_2, b_2)$ if and only if $a_1 \geq a_2$ and $b_1\mu(-a_2 + a_1) \geq b_2\mu(-a_2 + a_1)$.*

PROOF.

$$(a_1, b_1) - (a_2, b_2) = (a_1 - a_2, (b_1 - b_2)\sigma(-a_2)) \geq 0$$

if and only if $a_1 - a_2 \geq 0$ and

$$(b_1 - b_2)\sigma(-a_2)\mu(a_1 - a_2) = (b_1 - b_2)\mu(-a_2 + a_1)\sigma(-a_2) \geq 0.$$

But this is equivalent to $a_1 \geq a_2$ and $(b_1 - b_2)\mu(-a_2 + a_1) \geq 0$, which is the condition above. \square

LEMMA 3.9.

$$\begin{aligned} (a_1, b_1) \vee (a_2, b_2) &= (a_1 \vee a_2, b_1 \vee(-a_2 + a_1)^+ + b_2 \vee(-a_1 + a_2)^+ + (b_1 \vee b_2)\mu(-a_2 + a_1)), \\ (a_1, b_1) \wedge (a_2, b_2) &= (a_1 \wedge a_2, b_2 \vee(-a_2 + a_1)^+ + b_1 \vee(-a_1 + a_2)^+ + (b_1 \wedge b_2)\mu(-a_2 + a_1)). \end{aligned}$$

PROOF. Abbreviate by $(a_1 \vee a_2, b)$ the expression for $(a_1, b_1) \vee (a_2, b_2)$ above. We claim $(a_1 \vee a_2, b) \geq (a_1, b_1)$ for $a_1 \vee a_2 \geq a_1$ and

$$\begin{aligned} b\mu(-a_1 + a_1 \vee a_2) &= b\mu(-a_1 + a_2)^+ = b_1 \vee(-a_2 + a_1)^+ + (b_1 \vee b_2)\mu(-a_2 + a_1) \\ &\geq b_1 \vee(-a_2 + a_1)^+ + b_1 \mu(-a_2 + a_1). \end{aligned}$$

Now $A((-a_2 + a_1)^+) \vee A((-a_1 + a_2)^+) = A(-a_1 + a_2)$ and $A((-a_2 + a_1)^+) \cap A((-a_1 + a_2)^+) = 0$, and since π is a lattice homomorphism we have $[(-a_2 + a_1)^+] \boxplus [(-a_1 + a_2)^+] = [-a_1 + a_2]$ and $[(-a_2 + a_1)^+] \cap [(-a_1 + a_2)^+] = 0$, from which follows $[(-a_2 + a_1)^+] \boxplus [-a_1 + a_2]' = [(-a_1 + a_2)^+]'$. Therefore $b_1 \vee(-a_2 + a_1)^+ + b_1 \mu(-a_2 + a_1) = b_1 \mu(-a_1 + a_2)^+$, which completes the verification of the claim. A similar argument shows $(a_1 \vee a_2, b) \geq (a_2, b_2)$.

Suppose $(a_3, b_3) \geq (a_1, b_1), (a_2, b_2)$. Clearly $a_3 \geq a_1 \vee a_2$, and we know $b_3 \mu(-a_1 + a_3) \geq b_1 \mu(-a_1 + a_3)$ and $b_3 \mu(-a_2 + a_3) \geq b_2 \mu(-a_2 + a_3)$. We must show $b_3 \mu a \geq b \mu a$, where $a = -a_1 \vee a_2 + a_3 = (-a_1 + a_3) \wedge (-a_2 + a_3)$. Since $[(-a_2 + a_1)^+] \boxplus [(-a_1 + a_2)^+] \boxplus [-a_2 + a_1]' = B$, this amounts to showing that each term of $b_3 \mu a = b_3 \vee(-a_2 + a_1)^+ \mu a + b_3 \vee(-a_1 + a_2)^+ \mu a + b_3 \mu(-a_2 + a_1) \mu a$ exceeds (≥ 0) the corresponding term of $b \mu a$. We begin by showing

$$b_3 \vee(-a_2 + a_1)^+ \mu a \geq b_1 \vee(-a_2 + a_1)^+ \mu a.$$

Now $0 \leq (-a_2 + a_1)^+ = -a_2 + (a_1 \vee a_2) \leq -a_2 + a_3$, so $[(-a_2 + a_1)^+] \subseteq [-a_2 + a_3]$ or $[(-a_2 + a_2)^+] \cap [-a_2 + a_3]' = 0$, which together with $[-a_1 + a_3]' \vee [-a_2 + a_3]' = [a]'$ yields $[(-a_2 + a_1)^+] \cap [a]' \subseteq [-a_1 + a_3]'$. Because

$$b_3 \mu(-a_1 + a_3) \geq b_1 \mu(-a_1 + a_3)$$

it follows that $b_3 \vee(-a_2 + a_1)^+ \mu a \geq b_1 \vee(-a_2 + a_1)^+ \mu a$, and a similar argument shows $b_3 \vee(-a_1 + a_2)^+ \mu a \geq b_2 \vee(-a_1 + a_2)^+ \mu a$. It remains to show

$$b_3 \mu(-a_2 + a_1) \mu a \geq (b_1 \vee b_2) \mu(-a_2 + a_1) \mu a.$$

Now $A(-a_1 + a_3) \subseteq A(-a_1 + a_2) \vee A(-a_2 + a_3)$ and $A(-a_1 + a_3) \subseteq A(-a_1 + a_2) \vee A(-a_1 + a_3)$ imply $A(-a_1 + a_3) \subseteq A(-a_1 + a_2) \vee A(a)$, and similarly

$$A(-a_2 + a_3) \subseteq A(-a_1 + a_2) \vee A(a).$$

Therefore

$$[-a_1 + a_3] \vee [-a_2 + a_3] \subseteq [-a_1 + a_2] \vee [a],$$

or

$$[-a_1 + a_2]' \cap [a]' \subseteq [-a_1 + a_3]' \cap [-a_2 + a_3]'$$

Then $b_3\mu(-a_1 + a_3) \geq b_1\mu(-a_1 + a_3)$ and $b_3\mu(-a_2 + a_3) \geq b_2\mu(-a_2 + a_3)$ imply

$$b_3\mu(-a_2 + a_1)\mu a \geq (b_1 \vee b_2)\mu(-a_2 + a_1)\mu a. \quad \square$$

LEMMA 3.10. *B lies below A in G.*

PROOF. Given $0 < (0, b) \in B$ and $0 < (a, 0) \in A$, $(a, 0) \wedge (0, b) = (0, b\nu a)$, hence $n(a \wedge b) = (0, nb\nu a) \leq (a, 0)$ because $a \geq 0$ and $(nb\nu a)\mu a = 0 \leq 0\mu a$. \square

The reader may readily verify that $b\sigma(a) = -a + b + a$ and that $[a] = G(a) \cap B$ for each $a \in A$ and $b \in B$. Thus Lemma 3.10 completes the proof of Theorem 3.4. We shall refer to G as the splitting extension of B by A determined by σ and π . Because A is obviously maximal among subgroups of G lying above B and likewise B maximal below A , we know that A and B are closed l -subgroups of G , that B is convex, and that A is saturated.

The next few results concern $\mathcal{C}(G)$ as determined by $\mathcal{C}(A)$ and $\mathcal{C}(B)$. We continue to assume the terminology of Theorem 3.4.

PROPOSITION 3.11. *Let C and D be l-subgroups of A and B, respectively. Then $C \times D$ is an l-subgroup of G if and only if $D\sigma(c) = D$ and $D = (D \cap [c]) \boxplus (D \cap [c]')$ for all $c \in C$. In this case $C \times D$ is the splitting extension of D by C determined by (the restrictions of) σ and π .*

PROOF. Clearly $C \times D$ is closed under addition if and only if $D\sigma(c) = D$ for all $c \in C$. Given $0 < c \in C$ and $0 < d \in D$, $c \wedge d = (0, d\nu c)$. Thus in order for $C \times D$ to be a sublattice it is necessary that $D = (D \cap [c]) \boxplus (D \cap [c]')$ for all $c \in C$; Lemma 3.9 shows that this condition is sufficient as well. \square

Given an l -subgroup C of A let $B_C = \bigcup\{[c]: c \in C\}$. Notice that $B_C \in \mathcal{C}(B)$ because $\{[c]: c \in C\}$ is a directed set. Given the l -subgroup D of B let $A_D = \{a \in A: [a] \subseteq D\} \in \mathcal{C}(A)$.

PROPOSITION 3.12. *The following are equivalent for l-subgroups $C \leq A$ and $D \leq B$.*

- (a) $C \times D \in \mathcal{C}(G)$.
- (b) $C \in \mathcal{C}(A)$ and $B_C \subseteq D \in \mathcal{C}(B)$.
- (c) $D \in \mathcal{C}(B)$ and $A_D \supseteq C \in \mathcal{C}(A)$.

Furthermore, every $E \in \mathcal{C}(G)$ is of the form $E = C \times D$ for unique l -subgroups $C = E \cap A$ and $D = E \cap B$ satisfying the conditions above.

PROOF. Suppose $C \times D \in \mathcal{C}(G)$. Then $0 < x < c \in C$ implies $0 < (x, 0) < (c, 0) \in C \times D$, hence $(x, 0) \in C \times D$ or $x \in C \in \mathcal{C}(A)$. Likewise $D \in \mathcal{C}(B)$. For any $c \in C$ and $0 < b \in [c]$, $(0, 0) < (0, b) < (|c|, 0) \in C \times D$, hence $(0, b) \in C \times D$ by convexity, or $b \in D$. This proves that (a) implies (b); now assume (b) to prove (a). For $c \in C$, $D = [c] \boxplus (D \cap [c]')$ and $D\sigma(c) = [c]\sigma(c) \boxplus (D \cap [c]') = D$, so $C \times D$ is an l -subgroup by Proposition 3.11. Consider $0 < (x, y) < (c, d) \in C \times D$. We know from Lemma 3.8 that $0 \leq x \leq c$, $y\mu x \geq 0$, and $d\mu(-x + c) \geq y\mu(-x + c)$. It follows that $x \in C$ and that $d\mu c \geq y\mu c \geq 0$, and since $d\nu c \in [c] \subseteq D$, $d\mu c \in D$ and $y\mu c \in D$ by convexity. That is, $y = y\nu c + y\mu c \in D$, proving (a). (b) and (c) are clearly equivalent.

Suppose $E \in \mathcal{C}(G)$ and let $C = E \cap A$ and $D = E \cap B$. Observe that $0 < (x, y) \in G$ implies $0 \leq (x, 0) \leq 2(x, y) = (2x, y\sigma(x) + y)$ because $2x \geq x$ and

$$(y\sigma(x) + y)\mu(-x + 2x) = y\mu x \geq 0 = 0\mu(-x + 2x).$$

Therefore $0 < (c, d) \in E$ implies $(c, 0) \in E$ by convexity. If (c, d) is an arbitrary element of E , then $(c, d)^+ = (c^+, x) \in E$ and $(c, d)^- = (c^-, y) \in E$ for some $x, y \in B$, hence $(c^+, 0), (c^-, 0) \in E$ as above, and $(c, 0) = (c^+, 0) - (c^-, 0) \in E$ and $-(c, 0) + (c, d) = (0, d) \in E$, proving $E = C \times D$. \square

PROPOSITION 3.13. *The map $C \rightarrow C \times B$ is a lattice isomorphism from $\mathcal{C}(A)$ onto $\{E \in \mathcal{C}(G): B \subseteq E\}$. Thus C is a prime subgroup of A if and only if $C \times B$ is a prime subgroup of G . In particular, C is a value of $a \in A$ if and only if $C \times B$ is a value of $(a, b) \in G$ for any $b \in G$. In this case C is normal in its cover C^* if and only if $C \times B$ is normal in its cover $C^* \times B$.*

PROOF. The first assertion follows from Proposition 3.12, the second follows from the fact that a convex l -subgroup is prime if and only if the convex l -subgroups containing it form a chain, and the third and fourth assertions can be verified in straightforward fashion. \square

The following lemma is well known and quite general. We prove it here for completeness.

LEMMA 3.14. *Suppose B and D are convex l -subgroups of an l -group G and that $D \leq B$. Then $X = \{g \in G: |g| \wedge b \in D \text{ for all } b \in B^+\}$ is the maximum element of $\{E \in \mathcal{C}(G): E \cap B \subseteq D\}$. X is prime in G if and only if D is prime in B .*

PROOF. If $0 < x_1, x_2 \in X$ and $0 < b \in B$, then

$$\begin{aligned} 0 \leq b \wedge (x_1 + x_2) &\leq (2b) \wedge (b + x_2) \wedge (x_1 + b) \wedge (x_1 + x_2) \\ &= b \wedge x_1 + b \wedge x_2 \in D. \end{aligned}$$

It follows that X is a convex l -subgroup, and hence the maximum element of $\{E \in \mathcal{C}(G): E \cap B \subseteq D\}$. Suppose now that D is prime in B , and that $u, v \in G$ with $u \wedge v = 0$. If $u \notin X$, then there is some $b_0 \in B^+$ such that $u \wedge b_0 \notin D$. Thus for any $b \in B^+$, $u \wedge (b \vee b_0) \geq u \wedge b_0 \notin D$ and $[u \wedge (b \vee b_0)] \wedge [v \wedge (b \vee b_0)] = (u \wedge v) \wedge (b \vee b_0) = 0$ imply $v \wedge (b \vee b_0) \in D$, meaning $v \wedge b \in D$ or $v \in X$. \square

Given $B \in \mathcal{C}(G)$ let $\mathcal{C}_B(G) = \{X \in \mathcal{C}(G): X \subsetneq Y \in \mathcal{C}(G) \text{ implies } X \cap B \subsetneq Y \cap B\}$.

PROPOSITION 3.15. *The map $D \rightarrow A_D \times D$ is a one-one order preserving map from $\mathcal{C}(B)$ onto $\mathcal{C}_B(G)$. Thus D is a prime subgroup of B if and only if $A_D \times D$ is a prime subgroup of G , and every prime subgroup E of G such that $B \not\subseteq E$ has the form $E = A_D \times D$ for a (unique) proper prime subgroup D of B . For such a D , $A_D \times B$ is a value of (a, b) in G if and only if D is a value of $b\mu a$ in B .*

PROOF. The fact that $A_D \times D$ is the maximum element of $\{E \in \mathcal{C}(G): E \cap B \subseteq D\}$ follows from Proposition 3.12, and the second assertion results from Lemma 3.14. Suppose now that E is a prime subgroup of G for which $B \not\subseteq E$. By

Proposition 3.12 $E = C \times D$ for D a proper prime subgroup of B and $C \in \mathcal{C}(A)$ such that $C \leq A_D$. If $0 < a \in A_D \setminus C$, then $a \notin E$, so $[a]' \subseteq a^\perp \subseteq E$, hence $[a]' \subseteq E \cap B = D$. But together with the fact that $[a] \subseteq D$ since $a \in A_D$, this violates the assumption that $B \neq D$. We conclude that $C = A_D$ and $E = A_D \times D$.

Suppose that D is a proper prime subgroup of B , and fix $a \in A$ and $b \in B$. If D is a value of $b\mu a$ in B with cover D^* , then $(a, b) \notin A_D \times D$, and by the preceding paragraph every $E \in \mathcal{C}(G)$ satisfying $A_D \times D \subsetneq E$ also satisfies $D^* \subseteq E$, hence $A_D \times D$ is a value of $(0, b\mu a)$ in G . Now $(0, b\mu a)$ is a component of (a, b) (because $(a, b) = (a, b\nu a) + (0, b\mu a)$), so $A_D \times D$ is a value of (a, b) as well. Now suppose $A_D \times D$ is a value of (a, b) . It follows that $a \in A_D$, for otherwise $A_D \times B$ is a prime subgroup properly containing $A_D \times D$ which nevertheless does not contain (a, b) . By convexity, $(a, b\nu a) \in A_D \times D$; hence $(0, b\mu a) \notin D$, and if $D \subseteq E \in \mathcal{C}(B)$, then $(0, b\mu a) \in A_E \times E \supsetneq A_D \times D$, meaning that D is a value of $b\mu a$. \square

PROPOSITION 3.16. *If D and E are prime subgroups of B such that $D \vee E \neq B$ in $\mathcal{C}(B)$, then $A_D = A_E$.*

PROOF. It is sufficient to establish the case in which $D \subsetneq E \subsetneq B$. Now $A_E \times E$ and $A_D \times B$ are members of $\mathcal{C}(G)$ containing the prime subgroup $A_D \times D$, and so are related. Since $B \not\subseteq E$ we must have $A_E \times E \subseteq A_D \times B$, or $A_E \subseteq A_D$. \square

Proposition 3.16 completes the description of $\Gamma(G)$ from $\Gamma(A)$ and $\Gamma(B)$. Call a pair of prime subgroups D and E of B *distantly related* if $D \vee E \neq B$ in $\mathcal{C}(B)$ and let $\langle\langle E \rangle\rangle = \{D: E \text{ and } D \text{ are distantly related prime subgroups of } B\}$. Start with $\Gamma(A)$, and attach each equivalence class $\langle\langle E \rangle\rangle$ below $A_E \in \Gamma(A)$. (Actually, we are cheating a bit here. In this paragraph only are we considering $\Gamma(A)$ to be the partially ordered set of all (not only regular) primes of A .) The resulting partially ordered set is $\Gamma(G)$.

COROLLARY 3.17. *G is normal-valued if and only if both A and B are normal-valued.*

PROOF. Suppose that A and B are normal valued, that P is a regular prime of G with cover P^* , and that $(a, b) \in P^* \setminus P$. If $B \subseteq P$, then $P = C \times B$ for some value C of a in A having cover C^* . For any $(c, b_1) \in P$,

$$-(a, b) + (c, b_1) + (a, b) = (-a + c + a, b_2) \in C^* \times B = P^*.$$

If $B \not\subseteq P$, then $P = A_D \times D$ for some value D of $b\mu a$ in B having value D^* . Since $b\mu a$ is a component of (a, b) , $-(a, b) + P + (a, b) = -b\mu a + P + b\mu a$. So consider arbitrary $(c, d) \in P = A_D \times D$. Then

$$\begin{aligned} -(0, b\mu a) + (c, d) + (0, b\mu a) &= (c, -b\mu a\sigma(c) + d + b\mu a) \\ &= (c, -b\mu a\nu c\sigma(c) + (-b\mu a\mu c + d + b\mu a\mu c) + b\mu a\nu c), \end{aligned}$$

and $b\mu a\nu c\sigma(c)$ and $b\mu a\nu c$ lie in D because $[c] \subseteq D$ since $c \in A_D$, while $(-b\mu a\mu c + d + b\mu a\mu c) \in D$ because $b\mu a\mu c \in D^*$ since $b\mu a \in D^*$. Therefore $-(a, b) + P + (a, b) \subseteq P^*$. \square

Here is an example which shows that G need not lie in \mathbf{S} when A and B do. Let N be the trivially ordered set $\{0, 1, 2, \dots\}$, let $B = \Sigma(N, \mathbf{Z})$, let $A = \{g \in V(N, \mathbf{Z}): \text{there is some } m \in N \text{ such that } (n)g = (0)g \text{ for all } n \geq m\}$, and for each $a \in A$ let

$\sigma(a)$ be the identity l -automorphism of B and $\pi(a) = [a] = \{b \in B: (n)b \neq 0 \text{ implies } (n)a \neq 0 \text{ for all } n \in N\} \in \mathcal{S}(B)$. Let $a \in A$ be the element such that $(n)a = 1$ for all $n \in N$. Then no disjoint set of special elements in G can have supremum a .

COROLLARY 3.18. *G is finite-valued if and only if both A and B are.*

PROOF. The set of all values of $(a, b) \in G$ is $\{P \times B: P \text{ is a value of } a \text{ in } A\} \cup \{A_D \times D: D \text{ is a value of } b\mu a \text{ in } B\}$, a finite set. \square

We close this section by pointing out that the splitting extension developed in this section is general enough to encompass cardinal product, lex extension, and permutation wreath product.

THEOREM 3.19. *Given l -groups A and B define $\pi(A(a)) = [a] = 0 \in \mathcal{S}(B)$ for each $a \in A$ and define $\sigma(a)$ to be the identity automorphism of B . Then the splitting extension of B by A determined by π and σ is (l -isomorphic to) the cardinal product $A \boxplus B$.*

The notion of *lex extension* is developed in [10].

THEOREM 3.20. *Suppose σ is any group homomorphism from the totally ordered group A into the l -automorphisms of the l -group B , and suppose $\pi(A(a)) = [a] = B \in \mathcal{S}(B)$ for all $a \in A$. Then the splitting extension of B by A determined by π and σ is a lex extension of B by A . Conversely, if G is any l -group whose lex kernel B splits, that is for which $G = A + B$ for some subgroup A , then A is totally ordered and there is some homomorphism σ such that G is obtained from A and B as above.*

We employ the permutation wreath product notation of Chapter 5 of [13]. $\Pi_T G = \{f: f: T \rightarrow G\}$ designates the cardinal product of copies of G indexed by T . For $f \in \Pi_T G$ we designate the component of f at index t by f_t .

THEOREM 3.21. *Let (G, Ω) and (H, T) be transitive l -permutation groups and let $B = \Pi_T G$. For $h \in H$ define $[h] = \pi(H(h)) = \{f \in B: \alpha f_t = \alpha \text{ for all } \alpha \in \Omega \text{ and } t \in T \text{ such that } th = t\} \in \mathcal{S}(B)$, and define $f\sigma(h) \in B$ by $\alpha(f\sigma(h))_t = \alpha f_s$, where $s = th^{-1} \in T$. Then the splitting extension of B by H determined by σ and π is l -isomorphic to $G \text{ Wr } H$.*

4. Characterizing the lattice homomorphism $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(B)$. In some important instances it is possible to completely characterize the maps $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(B)$ of Proposition 3.3. The basic insight is lattice theoretical in nature.

PROPOSITION 4.1. *Suppose that D is a distributive lattice and E is a complete Boolean algebra. If for each prime ideal P of D there is an element $e_P \in E$ satisfying property (a) below, then the map $\pi: D \rightarrow E$ defined by $\pi(d) = \bigvee \{e_P: d \notin P\}$ enjoys properties (b), (c), and (d) below. Conversely, suppose $\pi: D \rightarrow E$ satisfies (b), (c), and (d) and define $e_P = (\bigwedge \{\pi(d): d \notin P\}) \wedge (\bigwedge \{\pi(d)^\perp: d \in P\})$ for P a prime ideal of D . Then (a) holds, and $\pi(d) = \bigvee \{e_P: d \notin P\}$ for all $d \in D$.*

- (a) $e_P \wedge e_Q = 0$ for distinct prime ideals P and Q .
- (b) π is a lattice homomorphism.

(c) $\pi(0) = 0$.

(d) *The complete subalgebra $F \subseteq E$ generated by $\pi(D)$ is atomic.*

PROOF. Suppose that for each prime ideal $P \subseteq D$ there is an element $e_P \in E$ satisfying (a), and define $\pi(d) = \bigvee \{e_P : d \notin P\}$. Let $A = \{e_P : P \text{ a prime ideal of } D\} \cup \{\bigwedge \{e_P^\perp : P \text{ a prime ideal of } D\}\}$, a maximal pairwise disjoint subset of E , and let $F = \{\bigvee A_0 : A_0 \subseteq A\}$, the complete subalgebra of E generated by A . Observe that F is isomorphic to the Boolean algebra of subsets of A , and that A is the set of atoms of F . Properties (b) and (c) are immediate consequences of these observations; to establish (d) we must show that $\pi(D)$ generates F . This is done by showing that $e_P = e$ for each prime ideal P of D , where e stands for $(\bigwedge \{\pi(d) : d \notin P\}) \wedge (\bigwedge \{\pi(d)^\perp : d \in P\})$. Now $e_P \leq e$ because e_P is an atom of F , and if $e_P < e$ it could only be because $e_Q \leq e$ for some prime ideal $Q \neq P$. But if $d \in Q \setminus P$, then $e_Q \leq \pi(d)^\perp$ while $e_P \leq \pi(d)$, and similarly for $d \in P \setminus Q$. This contradiction establishes that $e_P \in F$.

Suppose that $\pi: D \rightarrow E$ satisfies (b), (c), and (d) and define

$$e_P = \left(\bigwedge \{ \pi(d) : d \notin P \} \right) \wedge \left(\bigwedge \{ \pi(d)^\perp : d \in P \} \right)$$

for each prime ideal P of D . If P and Q are distinct prime ideals, then there is some $d \in D$ such that $d \in P \setminus Q$, say, and so $e_P \leq \pi(d)^\perp$ and $e_Q \leq \pi(d)$, proving (a). Now consider a particular $d \in D$ and let $e = \bigvee \{e_P : d \notin P\}$. Clearly $\pi(d) \geq e$; to show equality, consider an atom $a \in F$ such that $a \leq \pi(d)$. Let $Q = \{d' \in D : \pi(d') \not\geq d\} = \{d' \in D : \pi(d')^\perp \geq a\}$, a prime ideal of D . It is straightforward to verify that $a \leq e_Q \leq e$, from which follows $\pi(d) = e$. \square

The prime subgroups of an l -group A correspond to the prime ideals of the distributive lattice $\mathcal{C}\mathcal{H}(A)$. Therefore the following result is an application of the preceding one. In the following we use $[a]$ to designate $\pi(A(a))$.

THEOREM 4.2. *Let A and B be l -groups. If to each prime subgroup P of A there corresponds a polar $B_P \in \mathcal{P}(B)$ satisfying (a) and (b) below, then the map $\pi: \mathcal{C}\mathcal{H}(A) \rightarrow \mathcal{S}(B)$ defined by $[a] = \bigvee \{B_P : a \notin P\}$ enjoys properties (c), (d), and (e). Conversely, suppose $\pi: \mathcal{C}\mathcal{H}(A) \rightarrow \mathcal{S}(B)$ satisfies (c), (d), and (e), and for each prime subgroup P of A define $B_P = (\bigcap \{[a] : a \notin P\}) \cap (\bigcap \{[a]' : a \in P\})$. Then (a) and (b) hold, and $\pi(a) = \bigvee \{B_P : a \notin P\}$.*

(a) $B_P \cap B_Q = 0$ for distinct prime subgroups P and Q .

(b) $\bigvee \{B_P : a \notin P\} \in \mathcal{S}(B)$ for all $a \in A$. (The supremum here is taken in $\mathcal{P}(B)$.)

(c) π is a lattice homomorphism.

(d) $\pi(0) = 0$.

(e) *The complete subalgebra of $\mathcal{P}(B)$ generated by $\pi(\mathcal{C}\mathcal{H}(A))$ is atomic.*

LEMMA 4.3. *A complete subalgebra S of a complete atomic Boolean algebra T is itself atomic.*

PROOF. Let A be the set of atoms of T , and for each $a \in A$ let $(a) = \bigwedge \{s \in S : a \leq s\} \in S$. Then $B = \{(a) : a \in A\}$ is a maximal pairwise disjoint subset of T , hence $\{\bigvee B_0 : B_0 \subseteq B\}$ is a complete Boolean subalgebra of T contained in S . Since

$s = \bigvee \{a \in A: a \leq s\}$, it follows that $s = \bigvee \{(a) \in B: a \leq s\}$ for all $s \in S$. Therefore S is $\{\bigvee B_0: B_0 \subseteq B\}$ and B is the set of atoms of S . \square

COROLLARY 4.4. *Suppose A and B are l -groups and that B has a basis. Then any lattice homomorphism $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(B)$ such that $\pi(0) = 0$ is of the type described in Theorem 4.2.*

PROOF. B has a basis if and only if $\mathcal{P}(B)$ is atomic. \square

COROLLARY 4.5. *Suppose A and B are l -groups such that $\Gamma(A)$ is finite. Then any lattice homomorphism $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(B)$ such that $\pi(0) = 0$ has the form described in Theorem 4.2.*

PROOF. $\Gamma(A)$ is finite if and only if $\mathcal{C}_\#(A)$ is finite, in which case $\pi(\mathcal{C}_\#(A))$ generates a complete subalgebra of $\mathcal{P}(B)$ which is finite and so atomic. \square

COROLLARY 4.6. *Suppose B is any l -group and $A = \sum \mathbf{R}_\lambda$ is a small cardinal sum of archimedean o -groups. Then any lattice homomorphism $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(B)$ has the form described in Theorem 4.2.*

PROOF. Each \mathbf{R}_γ is an atom of $\mathcal{C}_\#(A)$ in the sense that it is minimal in $\mathcal{C}_\#(A) \setminus \{0\}$, and each element $A(a) \in \mathcal{C}_\#(A)$ is a finite join of such atoms. Therefore $\{\pi(\mathbf{R}_\gamma): \pi(\mathbf{R}_\gamma) > 0\}$ is the set of atoms of the complete subalgebra of $\mathcal{P}(B)$ generated by $\pi(\mathcal{C}_\#(A))$. \square

COROLLARY 4.7. *Suppose A and B are l -groups such that each positive element of B exceeds a positive special element. Then every lattice homomorphism $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(B)$ such that $\pi(0) = 0$ has the form described in Theorem 4.2.*

PROOF. Observe that for any special $b \in B^+$ and $a \in A$, either $b \in [a]$ or $b \in [a]'$. Let $(b) = (\bigcap \{[a]: a \in A \text{ and } b \in [a]\}) \cap (\bigcap \{[a]': a \in A \text{ and } b \in [a]'\})$, a nonzero element of $\mathcal{P}(B)$, and let $C = \{(b): 0 < b \in B, b \text{ special}\}$. Since each element of B exceeds a special element, $[a] = \bigvee \{(b) \in C: b \in [a]\}$ for each $a \in A$. Therefore the complete subalgebra of $\mathcal{P}(B)$ generated by $\pi(\mathcal{C}_\#(A))$ is atomic, and C is the set of its atoms. \square

If B is finite-valued, then the polars B_p of Theorem 4.2 are themselves cardinal summands. To prove this it is necessary only to observe that $\mathcal{S}(B)$ is a complete subalgebra of $\mathcal{P}(B)$ in this case.

LEMMA 4.8. *If G is finite-valued, then $\mathcal{S}(G)$ is a complete sublattice of $\mathcal{P}(G)$.*

PROOF. Observe that for special $t \in G^+$ and $C \in \mathcal{S}(G)$, $t \in C$ or $t \in C^\perp$. ($t = c + d$ for $c \in C$ and $d \in C^\perp$, and the value P of t contains exactly one of c and d , say d . For every prime subgroup $Q \subseteq P$, $c \notin P$ implies $(t - c) = d \in Q$, or $Q + t = Q + c$. For any prime $Q \not\subseteq P$ we get $t \in Q$, since some multiple of $q \in Q \setminus P$ will exceed t . Therefore $Q \leq Q + c \leq Q + t = Q$ in that case also. This proves $t \in c$.) Suppose now that $\bigvee \mathcal{C} = D$ in $\mathcal{P}(B)$ for some $\mathcal{C} \subseteq \mathcal{S}(B)$, and consider $g \in G^+$. Then $g = g_1 + g_2 + \cdots + g_n$ for disjoint special elements g_i . By relabelling we may assume $g_i \in \bigcup \mathcal{C}$ for $1 \leq i \leq k$ and $g_i \notin \bigcup \mathcal{C}$ for $k < i \leq n$.

Then $(g_1 + g_2 + \cdots + g_k) \in D$ and $(g_{k+1} + g_{k+2} + \cdots + g_n) \in \bigcap \{C^\perp : C \in \mathcal{C}\} = D^\perp$. \square

PROPOSITION 4.9. *Suppose A and B are l -groups such that B is finite-valued, and that $\pi: \mathcal{C}_\ell(A) \rightarrow \mathcal{S}(B)$ is a lattice homomorphism with $\pi(0) = 0$. Then for each prime P of A there is a summand $B_P \in \mathcal{S}(B)$ such that $B_P \neq B_Q$ for distinct primes P and Q , and such that $[a] = \sum \{B_P : a \notin P\}$ for each $a \in A$.*

The following theorem summarizes the results of this investigation as applied to lattice orderings of semidirect products of l -groups in which the factors appear above and below one another.

THEOREM 4.10. *Suppose A and B are l -groups, and suppose the group semidirect product $G = A \times B$ is endowed with a lattice order extending the orders on A and B . If A lies above B , then A is maximal above B and hence is saturated, and B is maximal below A and hence convex. Conversely, if A is saturated, B is convex, and if B satisfies (a) or (b) below, then A lies above B . Furthermore, if A lies above B and any one of (c), (d), (e), or (f) holds, then the lattice order on G is determined as per Theorem 4.2.*

- (a) $B = \vee(T \wedge b)$ for all $b \in B^+$, where $T = \{t \in B^+ : t \text{ special}\}$.
- (b) $\mathcal{S}(B)$ is dense in $\mathcal{P}(B)$.
- (c) B has a basis.
- (d) $\Gamma(A)$ is finite.
- (e) A is a cardinal sum of archimedean o -groups.
- (f) Each positive element of B exceeds a positive special element.

PROOF. The subgroup A is maximal with respect to $A \cap B = 0$, and so is maximal above B whenever it lies above B . In this case A must be saturated and B convex by Theorem 1.6. The converse is Proposition 1.12. The last assertion has been established in the preceding corollaries. \square

As an example of Theorem 4.10 consider an l -group G which is a splitting extension of the l -ideal B by the saturated l -subgroup A , where B is l -isomorphic to $\Sigma_{i=1}^n \mathbb{Z}_i$ and A is divisible. Since the group of l -automorphisms of B is the group of permutations on n elements and since any homomorphic image of A is divisible, G is the group direct product of A and B . Furthermore, B lies below A , and the order on G is given by assigning a cardinal summand of B to each prime of A as in Theorem 4.2.

5. l -groups in which each maximal below subgroup splits. Let $S_1(S_2)$ be the class of l -groups G such that if B is a maximal below subgroup of G , then $G = A + B$ for some subgroup (every subgroup) A maximal above B . We shall show that these classes are closed with respect to convex l -subgroups, cardinal sums, and cardinal products.

LEMMA 5.1. *Suppose (A, B) is a maximal above and below pair of $G = G_1 \boxplus G_2$. Then $B = (B \cap G_1) \boxplus (B \cap G_2)$ and $A = (A \cap G_1) \boxplus (A \cap G_2)$, so $(A \cap G_i, B \cap G_i)$ is a maximal above and below pair in G_i for $i = 1, 2$.*

PROOF. $0 < b \in B$ implies $b = b_1 + b_2$ for unique $b_i \in G_i$, and $b_1, b_2 \geq 0$ because the order on G is the lattice product of the orders on the G_i 's. Therefore $b_1, b_2 \in B$ by convexity, meaning $B = (B \cap G_1) \boxplus (B \cap G_2)$. Likewise $0 < a \in A$ implies $a = a_1 + a_2$ for unique $a_i \in G_i$, and the a_i 's are components of a , again by virtue of the order on G . Then $a_1, a_2 \in A$ because A is saturated by Theorem 1.6, so that $A = (A \cap G_1) \boxplus (A \cap G_2)$.

It is clear from Proposition 1.4 that $A \cap G_1$ lies below $B \cap G_1$ in G_1 . If C is maximal among those convex l -subgroups of G_1 which contain $A \cap G_1$ and lie below $B \cap G_1$, then $C \boxplus (B \cap G_2)$ lies below A , hence $C = A \cap G_1$. A similar argument shows that $A \cap G_1$ is maximal among l -subgroups of G_1 above $B \cap G_1$. \square

PROPOSITION 5.2. *If $G \in \mathbf{S}_n$ and $C \in \mathcal{C}(G)$, then $C \in \mathbf{S}_n$, $n = 1, 2$.*

PROOF. Suppose $G \in \mathbf{S}_1$, $C \in \mathcal{C}(G)$, and B is maximal below A in C . Find $B_1 \in \mathcal{C}(G)$ maximal among those convex l -subgroups of G containing B and lying below A . Note that $B_1 \cap C$ lies below A in C , hence $B_1 \cap C = B$. Now find A_1 maximal among l -subgroups of G above B_1 such that $G = A_1 + B_1$. By Proposition 3.12 $C = (A_1 \cap C) + B$, proving $C \in \mathbf{S}_1$. The argument for $G \in \mathbf{S}_2$ is similar. \square

PROPOSITION 5.3. *The following are equivalent for any set $\{G_\lambda: \lambda \in \Lambda\}$ of l -groups.*

- (a) $G_\lambda \in \mathbf{S}_1$ for all $\lambda \in \Lambda$.
- (b) $\Sigma G_\lambda \in \mathbf{S}_1$.
- (c) $\Pi G_\lambda \in \mathbf{S}_1$.

These conditions remain equivalent if \mathbf{S}_1 is replaced by \mathbf{S}_2 .

PROOF. Proposition 5.2 provides the implication from (c) to (b) to (a). Suppose that $G_\lambda \in \mathbf{S}_1$ for all $\lambda \in \Lambda$ and let B be maximal below A in ΠG_λ . We may without loss of generality assume that A is maximal above B , so that by Lemma 5.1 $B \cap G_\lambda$ is maximal below $A \cap G_\lambda$ in G_λ for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ choose A_λ maximal above $B \cap G_\lambda$ such that $G_\lambda = (B \cap G_\lambda) + A_\lambda$, and let $A_1 = \{f \in \Pi A_\lambda: \lambda f \in A_\lambda \text{ for all } \lambda \in \Lambda\}$. The componentwise nature of the operations on ΠG_λ makes it clear that A_1 lies above B and that $A_1 + B = \Pi G_\lambda$. The argument for \mathbf{S}_2 is similar. \square

PROPOSITION 5.4. *If G is an archimedean l -group, then $G \in \mathbf{S}_2$ if and only if $G \in \mathbf{S}_1$ if and only if G is projectable.*

PROOF. In an archimedean l -group the only maximal above and below pairs are complementary polars. \square

The most general condition sufficient for membership in \mathbf{S}_2 is given by Theorem 6.18 below, which we state here. Its proof in §6 depends on none of the intervening material.

THEOREM 5.5. *A divisible abelian sup \mathcal{V} complete l -group lies in \mathbf{S}_2 .*

In the following proposition let $F(\Gamma, \mathbf{R})$ denote $\{g \in V(\Gamma, \mathbf{R}): \{\gamma \in \Gamma: (\gamma)g \neq 0\} \text{ admits only finite disjoint subsets}\}$ [9]. Observe that $\Sigma \leq F \leq V$.

PROPOSITION 5.6. *For any root system Γ , $\Sigma(\Gamma, \mathbf{R}) \in \mathbf{S}_1$, $F(\Gamma, \mathbf{R}) \in \mathbf{S}_1$, and $V(\Gamma, \mathbf{R}) \in \mathbf{S}_2$; furthermore $\Sigma(\Gamma, \mathbf{R}) \in \mathbf{S}_2$ and $F(\Gamma, \mathbf{R}) \in \mathbf{S}_2$ whenever Γ satisfies the DCC.*

PROOF. The fact that $V(\Gamma, \mathbf{R}) \in \mathbf{S}_2$ is a consequence of Theorem 5.5. To see that $\Sigma(\Gamma, \mathbf{R}) \in \mathbf{S}_1$ consider $B \in \mathcal{C}(\Sigma)$. Let $A = \{g \in \Sigma: \gamma g = 0 \text{ for all } \gamma \in \Gamma(B)\}$, an l -subgroup which lies above B by Proposition 1.4. Since $B = \{g \in \Sigma: \gamma g = 0 \text{ for all } \gamma \notin \Gamma(B)\}$, $A + B = \Sigma$. The proof that $F \in \mathbf{S}_1$ is analogous. Now suppose Γ has the DCC and that (A, B) is a maximal above and below pair in F . (The argument that $\Sigma \in \mathbf{S}_2$ is analogous.) Suppose for contradiction that $g \in F \setminus (A + B)$. At least one of the (finitely many) special components of g —say g_1 —fails to lie in $A + B$. Let γ_1 be the value of g_1 . Corollary 1.11 provides $a \in A$ such that $g_1 - a$ has a value in $\Gamma(B)$. This implies that one of the special components of a —say a_1 —has value γ_1 and that $(\gamma_1)a_1 = (\gamma_1)g_1$. Now $a_1 \in A$ because A is saturated, so $g_1 - a_1 \notin A + B$. Let g_2 be one of the special components of $g_1 - a_1$ such that $g_2 \notin A + B$, and let γ_2 be its value. Observe that $\gamma_2 < \gamma_1$. But a continuation of this argument produces an infinite descending chain in Γ , contrary to the DCC. \square

The following example makes several points, the first being that $\Sigma(\Gamma, \mathbf{R})$ need not lie in \mathbf{S}_2 in general. Let Γ be $\Delta \cup \Lambda$, where $\Delta(\Lambda) = \{0^n (0^n 1): n = 1, 2, 3, \dots\}$. Here $0^n (0^n 1)$ refers to the string of n zeros (followed by a single 1). Order Γ by declaring $\alpha > \beta$ whenever $\beta = \alpha\gamma$ for some $\gamma \in \Gamma$ (where $\alpha\gamma$ is the string formed by concatenating the α and γ strings). Let $B = \{g \in \Sigma(\Gamma, \mathbf{R}): \Gamma(g) \subseteq \Lambda\}$ and define $a_n \in \Sigma$ by

$$(\gamma)a_n = \begin{cases} 1 & \text{if } \gamma = 0^n, 0^n 1, \text{ or } 0^{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Let A be the l -subspace of the vector lattice Σ generated by $\{a_n: n = 1, 2, 3, \dots\}$. Then A lies above B by Proposition 1.4 since $\Gamma(A) \subseteq \Delta$, and in fact (A, B) is a maximal above and below pair by Corollary 1.11. Yet $A + B$ is proper in Σ , since the latter fails to contain the element $g \in \Sigma$ defined by

$$(\gamma)g = \begin{cases} 1 & \text{if } \gamma = 0, \\ 0 & \text{otherwise.} \end{cases}$$

A finite-valued vector lattice need not belong to \mathbf{S}_1 . Let Γ have the meaning above. In his thesis [8], Byrd showed that $\Sigma(\Gamma, \mathbf{R})$ has an uncountable number of pairwise non- l -isomorphic a -closures in the class of abelian l -groups, each of which is a vector lattice G which may be taken to satisfy $\Sigma(\Gamma, \mathbf{R}) \leq G \leq V(\Gamma, \mathbf{R})$. (See Chapter 4 of [9] for additional background.) We will show that exactly one of these a -closures lies in \mathbf{S}_1 . Now if B has the meaning above then it is still maximal below A in G ; therefore if $G \in \mathbf{S}_1$ then there must be some l -subgroup A_1 above B in G such that $G = A_1 + B$. The map $\theta: A_1 \rightarrow V(\Gamma, \mathbf{R})$ defined by

$$(\gamma)\theta(a) = \begin{cases} (\gamma)a & \text{if } \gamma \in \Delta, \\ 0 & \text{if } \gamma \in \Lambda, \end{cases}$$

is an l -monomorphism and so it can be extended to an l -monomorphism $\hat{\theta}: G \rightarrow V$, which must be the identity on B . (See Theorem 3.3 of [12].) Since $G\hat{\theta} \leq F(\Gamma, \mathbf{R}) = \{g \in V: \{\gamma \in \Lambda: (\gamma)g \neq 0\} \text{ finite}\}$ and since F is an a -extension of Σ , it follows that G is l -isomorphic to F .

COROLLARY 5.7. *A finite-valued vector lattice G of countable dimension belongs to S_1 .*

PROOF. G is l -isomorphic to $\Sigma(\Gamma, \mathbf{R})$ for a suitable choice of Γ [9]. \square

PROPOSITION 5.8. *A totally ordered group G belongs to S_1 if and only if each convex subgroup splits. That is, for each $B \in \mathcal{C}(G)$ there is some subgroup A such that $A + B = G$.*

PROOF. Suppose $G \in S_1$. Each regular subgroup of G is maximal below by Proposition 2.3 and so splits. In particular, each regular subgroup and hence each convex subgroup is normal. Given $B \in \mathcal{C}(G)$ choose $a_\gamma \in G^\gamma \setminus G_\gamma$ for each regular G_γ such that $G_\gamma \supseteq B$. Then B is maximal below the subgroup A generated by the a_γ 's. The converse is obvious. \square

PROPOSITION 5.9. *Suppose $G \in S_2$. If (A_1, B) and (A_2, B) are maximal above and below pairs in G , then A_1 is l -isomorphic to A_2 .*

PROOF. Both A_1 and A_2 are l -isomorphic to G/B . \square

Does Proposition 5.9 hold even when G does not lie in S_2 ? The authors know of no counterexample to this conjecture. However, the following example provides counterexamples to several closely related conjectures. Let $\Gamma = \{a_1, a_2, b_1, b_2\}$, ordered by declaring $a_i > b_i$, and let $B = V(\Gamma, \mathbf{Z})$. Let A be \mathbf{Z} , $\pi(A(a)) = [a] = B \in \mathcal{S}(B)$ for all $0 \neq a \in A$, and let $\sigma(1)$ be the l -automorphism of B defined by $(x)(g\sigma(1)) = (\psi(x))g$, where $\psi(a_1) = a_2$, $\psi(a_2) = a_1$, $\psi(b_1) = b_2$, and $\psi(b_2) = b_1$. Let G be the splitting extension of B by A determined by π and σ . G is a Scrimger group [15] and so normal-valued, and its elements can be naturally represented in the form

$$\begin{pmatrix} & m & \\ n_1 & & n_2 \\ p_1 & & p_2 \end{pmatrix}$$

for integers m , n_i , and p_i . Using this notation let $A_1 = \{g \in G: m = n_1 = p_1 = p_2 = 0\}$, $A_2 = \{g \in G: m \text{ even, } n_1 = p_1 = p_2 = 0\}$, $A_3 = \{g \in G: p_1 = p_2 = 0\}$, $B_1 = \{g \in G: m = n_1 = n_2 = p_2 = 0\}$, $B_2 = \{g \in G: m = n_2 = 0\}$, and $B_3 = \{g \in G: m = n_1 = n_2 = 0\}$. Observe that B_1 lies below A_1 , that B_2 is maximal with respect to containing B_1 and lying below A_1 , that A_2 is maximal with respect to containing A_1 and lying above B_2 , that A_3 is maximal with respect to containing A_1 and lying above B_1 , and that B_3 is maximal with respect to containing B_1 and lying below A_3 . The point is that A_2 is not l -isomorphic to A_3 , B_2 is not l -isomorphic to B_3 , and that $A_2 + B_2$ is not l -isomorphic to $A_3 + B_3$.

6. Abelian l -groups G which split off from below in every abelian extension in which G is a convex l -subgroup. Suppose G is a convex l -subgroup of the l -group H . We say G *splits off from below* in H if $H = U + G$ for every l -subgroup $U \leq H$ such that U is maximal above G . The subject of this section is a characterization of those abelian l -groups G which split off from below in every abelian l -group having G as a convex l -subgroup (Theorem 6.16). If G has this property we shall sometimes simply say that G splits off from below. The following lemma is useful; it is Lemma 5.6 of [2].

LEMMA 6.1. *Suppose X is a subset and g an element of G such that $g \geq X$. Then X is type \mathcal{Y} and $\bigvee X = g$ if and only if $\bigvee \{(y - x)^\perp : x \in X\} = 1_\emptyset$.*

An l -group G which splits off from below is closed in every l -group H in which G is a convex l -subgroup. The latter property depends on the possession by G of a supremum for each subset of the following type. Let us call a subset $X \subseteq G^+$ type \mathcal{Z} if X is type \mathcal{Y} and if $\bigvee (X \wedge g)$ exists in G for all $g \in G$. G is *sup \mathcal{Z} complete* if every type \mathcal{Z} subset of G possesses a supremum in G .

PROPOSITION 6.2. *If G is sup \mathcal{Z} complete, then G is closed in every l -group H in which G is a convex l -subgroup.*

PROOF. Suppose $\bigvee X = h \in H$ for $X = G^+ \wedge h \subseteq G^+$. We first claim that X is a type \mathcal{Y} subset of H^+ . Lemma 6.1 is enough to show $\bigvee_{x \in X} (h - x)^\perp = 1_\emptyset$ in the Boolean algebra \mathcal{P} of polars of H . For this purpose consider $0 < g \in H$; we shall prove that $g \notin (h - x)^\perp$ for some $x \in X$. Since $G^{\perp\perp}$ is closed in H it follows that $h \in G^{\perp\perp}$, which allows us to consider only the case in which $g \in G$. Because $\bigvee X = h$, there must be some $x \in X$ with $x \not\leq -g + h$ or $g + x \not\leq h$. Let $x' = (g + x) \wedge h \in X$; we claim $g \notin (h - x')^\perp$. For $h \vee (g + x) > h$ implies $0 < (h \vee (g + x)) - h = 0 \vee (g + x - h) = g' \leq g$, and $(h - x') \wedge g' = [0 \vee (h - x - g)] \wedge [0 \vee (g + x - h)] = 0 \vee (-|h - x - g|) = 0$. This completes the proof that X is a type \mathcal{Y} subset of H . Since G is convex in H , intersection provides a one-one correspondence from $\{P \in \mathcal{P} \mid P \subseteq G^{\perp\perp}\}$ onto the set of polars of G , from which easily follows that X is a type \mathcal{Y} subset of G . X is actually type \mathcal{Z} , since by convexity $\bigvee (X \wedge g) = (\bigvee X) \wedge g = h \wedge g \in G$ for any $g \in G$. Since G is sup \mathcal{Z} complete there is some t such that $\bigvee X = t \in G$. But then $0 \leq h \leq t$ implies by convexity that $h \in G$. \square

COROLLARY 6.3. *If G is sup \mathcal{Y} complete, then it is closed in any l -group in which it is a convex l -subgroup.*

THEOREM 6.4. *An abelian l -group G is closed in every l -group in which it is a convex l -subgroup if and only if it is sup \mathcal{Z} complete.*

PROOF. Let $G_{\mathcal{Y}}$ be the sup \mathcal{Y} completion of G [1] and suppose $X \subseteq G^+$ is a type \mathcal{Z} set with $\bigvee X = t \in G_{\mathcal{Y}} \setminus G$, and let W be the l -subgroup of $G_{\mathcal{Y}}$ generated by G and t . We shall prove that G is convex but not order closed in W . For this purpose we shall adopt the following terminology for this proof and the next one only. An element $s \in G$ will be *admissible* if $s\pi_0 g \in G$ for all $0 < g \in G$, where $\pi_{ab}: G_{\mathcal{Y}} \rightarrow [a, b]$ is

the lattice homomorphism defined by $x\pi ab = (x \vee a) \wedge b$. Observe that G is \mathcal{Z} complete if and only if it contains each admissible element of $G_{\mathcal{Z}}$. However, the set of admissible elements is not closed under addition in general.

The first claim is that if $0 < u, v \in G_{\mathcal{Z}}$ are admissible, then so is $u + v$. This is true because

$$\begin{aligned} (u + v)\pi 0g &= (u + v) \wedge g = (u + v) \wedge (g + v) \wedge (u + g) \wedge (g + g) \wedge g \\ &= (u \wedge g + v \wedge g) \wedge g = (u\pi 0g + v\pi 0g)\pi 0g \in G. \end{aligned}$$

Therefore nt is admissible for all positive integers n . The second claim is that if $0 < u \in G_{\mathcal{Z}}$ is admissible, then $u\pi ab \in G$ for all $a < b$ in G . This is true because

$$u\pi ab = (u\pi ab)^+ + (u\pi ab)^- = u\pi a^+b^+ + b^- = u\pi 0b^+ \vee a^+ + b^- \in G.$$

The third claim is that if $0 < u \in G_{\mathcal{Z}}$ is admissible, then $(k + u)\pi ab \in G$ for all $k, a, b \in G$ with $a < b$. This holds because

$$(k + u)\pi ab = k + [(u \vee (-k \vee a)) \wedge (-k \vee b)],$$

the latter term being in G by the second claim. The fourth and final claim is that if $0 < u \in G_{\mathcal{Z}}$ is admissible, then

$$(k - u)\pi ab = -(-k + u)\pi ab = -[(-k + u)\pi(-b)(-a)] \in G$$

for all $k, a, b \in G$ with $a < b$.

To complete the proof of the theorem consider an arbitrary $w \in W$. w has the form $\bigvee_{i=1}^n \bigwedge_{j=1}^m (k_{ij} + n_{ij}t)$ for $k_{ij} \in G$ and integers n_{ij} . The claims show each $k_{ij} + n_{ij}t$ to be admissible, and hence w is admissible. Therefore G is convex in W . However, G is clearly not order closed because $\bigvee X = t \in W \setminus G$. \square

COROLLARY 6.5. *Given any abelian l -group G there is an abelian l -group H in which G is an order dense convex l -subgroup and such that H is closed in any l -group containing H as a convex l -subgroup.*

PROOF. Let $G_0 = G$. Having defined G_α , define $G_{\alpha+1}$ to be G_α if G_α is \mathcal{Z} complete, otherwise let $G_{\alpha+1}$ be an l -subgroup of $G_{\mathcal{Z}}$ generated by G_α and some admissible $0 < t \in G_{\mathcal{Z}} \setminus G$. Take unions at limit ordinals, and let $H = G_\alpha$ for the least ordinal α such that $G_\alpha = G_{\alpha+1}$. The proof of Theorem 6.4 shows that G is a convex l -subgroup of H . \square

It seems quite unlikely that the l -group H of the preceding corollary is unique. The general question is open: which l -groups admit unique \mathcal{Z} hulls?

Those abelian l -groups G which split off from below in every abelian l -group having G as a convex l -subgroup can be characterized by the possession of three properties (Theorem 6.16). One of these properties is divisibility.

PROPOSITION 6.6. *If an abelian l -group G splits off from below in every lex extension, then G is divisible.*

PROOF. Let G^d be the divisible hull of G and suppose p is the least positive integer such that $px \in G$ for some $0 < x \in G^d \setminus G$. Consider the lexicographic product $\mathbf{Z} \vec{\times} G^d$ determined by letting $\sigma(n)$ be the identity l -automorphism on G^d

and $\pi(n) = [n] = G^d$ for each $n \in \mathbf{Z}$. Let H be the l -subgroup consisting of all pairs of the form $(pi + j, a + jx)$, where $i, j \in \mathbf{Z}$ and $a \in G$. Such pairs form an l -subgroup because $(pi + j, a + jx) \vee (0, 0)$ is either $(pi + j, a + jx)$ in case $pi + j > 0$, or is $(0, 0)$ in case $pi + j < 0$, or is $(0, (a - i(px)) \vee 0)$ in case $pi + j = 0$. H has (a copy of) G as the convex l -subgroup $0 + G$. Let $U = \{(pi, a) | i \in \mathbf{Z}\}$, an l -subgroup of H above G . To show that U is maximal above G consider an arbitrary $0 < h = (pi + j, a + jx) \in H \setminus U$ and let V be the l -subgroup of H generated by U and h . We may assume $0 \leq j < p$, and by subtracting $(pi, 0) \in U$ we may assume $h = (j, a + jx)$. If $j = 0$ then $a \neq 0$, so h itself has a value in $\Gamma(G)$, and if $j > 0$ then $ph - (pj, 0) = (0, pa + j(px)) \in V$ which also has a value in $\Gamma(G)$ since $0 \neq pa + j(px) \in G$. This proves U maximal above G , which proves the theorem since $U + G$ fails to contain $(1, x)$, among others. \square

Another characteristic of an abelian l -group G which splits off from below is the *summand sewing property* or SSP. Given a collection \mathcal{S} of summands of G , a map $\theta: \mathcal{S} \rightarrow G$ is said to be *consistent* if $S\theta - R\theta \in (S \cap R)^\perp$ for all $S, R \in \mathcal{S}$. G has the summand sewing property if for every consistent $\theta: \mathcal{S} \rightarrow G$ there is some $g \in G$ such that $g - S\theta \in S^\perp$ for all $S \in \mathcal{S}$. Observe that such $g \in G$ must always exist for finite \mathcal{S} . Also note that g need not be unique. Finally, observe that it is enough to consider consistent maps $\theta: \mathcal{S} \rightarrow G^+$.

The definition of the SSP just given bears close resemblance to the definition of orthocompleteness given in [2], and in fact this resemblance can be converted with little difficulty to a proof that orthocompleteness implies the SSP. Thus every l -group is order dense in an l -group which enjoys the SSP.

PROPOSITION 6.7. *If G is sup \mathcal{U} complete, then G has the SSP.*

PROOF. Suppose $\theta: \mathcal{S} \rightarrow G^+$ is a consistent map. For each $S \in \mathcal{S}$ let $g_S \in G$ be the projection of $S\theta$ on S , and let X be $\{g_S | S \in \mathcal{S}\}$. The fact that $(X - g_S)^+ \cup (-g_S + X)^+ \subseteq S^\perp$ for all $S \in \mathcal{S}$ demonstrates that X is a type \mathcal{U} set and that $\bigvee X = g$ exists in G . But it can readily be shown that $g - S\theta \in S^\perp$ for all $S \in \mathcal{S}$. \square

PROPOSITION 6.8. *An abelian l -group G which splits off from below in every abelian l -group in which it is maximal below must have the SSP.*

PROOF. Let $\mathcal{S} = \{S_\lambda: \lambda \in \Lambda\}$ be a set of summands of G and $\theta: \mathcal{S} \rightarrow G^+$ a consistent map which denies the SSP. Let \bar{G} be the orthocompletion of G and choose $t \in \bar{G}$ such that $t - S_\lambda\theta \in S_\lambda^\perp$ for all $\lambda \in \Lambda$. Let $\prod \mathbf{Z}_\lambda$ and $\sum \mathbf{Z}_\lambda$ designate as usual the large and small cardinal products of the integers \mathbf{Z} over the index set Λ . Designate by $c \in \prod \mathbf{Z}$ the element defined by $\lambda c = 1$ for all $\lambda \in \Lambda$, and let $A = \{nc + s \in \prod \mathbf{Z}: n \in \mathbf{Z}, s \in \sum \mathbf{Z}\}$, an l -subgroup of $\prod \mathbf{Z}$. Define $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(\bar{G})$ by declaring $\pi(A(a)) = [a] = \bigvee \{\bar{S}_\lambda: \lambda \in \text{supp}(a)\}$, where $\bar{S}_\lambda = S_\lambda^{\perp\perp} \in \mathcal{S}(\bar{G})$. Observe that for any $a \in A$, $\text{supp}(a)$ is a finite or cofinite subset of Λ and so $\bigvee \{\bar{S}_\lambda: \lambda \in \text{supp}(a)\}$ exists in $\mathcal{S}(\bar{G})$. Let σ be the map which assigns the identity l -automorphism of \bar{G} to each $a \in A$, and let K be the splitting extension of \bar{G} by A determined by σ and π . Finally, let H be $\{(nc + s, nt + g) \in K: n \in \mathbf{Z}, s \in \sum \mathbf{Z}, g \in G\}$.

We first claim that H is an l -subgroup of K . To verify this claim we must succeed in showing $(nc + s, nt + g) \vee (0, 0)$ lies in H . If $n = 0$ then $\text{supp}(nc + s) = \text{supp}(s)$ is finite, and $(s, g) \vee (0, 0) = (s \vee 0, g')$, where $g' = g \vee 0 + \sum\{g_\lambda^- : \lambda \in \text{supp}(s^+)\} - \sum\{g_\lambda : \lambda \in \text{supp}(s^-)\} \in G$. Here g_λ designates the element of S_λ such that $g - g_\lambda \in S_\lambda^\perp$. If $n > 0$ then $\text{supp}(nc + s)$ is cofinite, $\text{supp}((nc + s)^-)$ is finite, and

$$(nc + s, nt + g) \vee (0, 0) = ((nc + s) \vee 0, g'),$$

where

$$g' = g - \sum\{(nt_\lambda + g_\lambda)^- : \lambda \notin \text{supp}(nc + s)\} \\ - \sum\{(nt_\lambda + g_\lambda) : \lambda \in \text{supp}((nc + s)^-)\} \in G.$$

Here t_λ is the unique element of S_λ such that $S_\lambda\theta - t_\lambda \in S_\lambda^\perp$. The case in which $n < 0$ is handled similarly.

Let $U = \{(u, 0) : u \in \sum \mathbb{Z}\}$, an l -subgroup of H which lies above G . To show that U is maximal above G consider $h = (nc + u, nt + g) \in H \setminus U$ and let W be the l -subgroup of H generated by $U \cup \{h\}$. Now for each $\lambda \in \Lambda$ there is an element $u_\lambda \in \sum \mathbb{Z}$ such that $(\lambda)(nc + u + u_\lambda) = 0$. Therefore $nt_\lambda + g_\lambda \in G$ is a component of $h + (u_\lambda, 0) \in W$. Now if $nt_\lambda + g_\lambda = 0$ for all λ , then $nt + g \in S_\lambda^\perp$ for all λ , hence $t - f \in S_\lambda^\perp$ for all λ , where f is the element proven to exist in G by Proposition 6.6 such that $n(-f) = g$. This contradicts the assumption that θ violates the SSP and proves that $nt_\lambda + g_\lambda \neq 0$ for some $\lambda \in \Lambda$. But then $\Gamma(W) \cap \Gamma(G) \neq \emptyset$, proving that U is maximal above G . This proves the theorem since $U + G$ fails to contain $(c, t) \in H$, among others. \square

Suppose P is a polar of the l -group G . We shall say that a set $X \subseteq G^+$ is of type $\mathcal{W}(P)$ if X is type \mathcal{U} , $X \subseteq P^+$, and for every $0 \leq g \in G$ there is some element $g' \in G$ satisfying the following three conditions: $g - g' \in P$, $X \wedge g \leq g'$, and $P \subseteq \bigvee\{(g' - x \wedge g)^\perp : x \in X\}$. We shall call G *sup \mathcal{W} complete* if $\forall X$ exists in G for every polar P and every $X \subseteq G^+$ of type $\mathcal{W}(P)$. Several comments deserve mention here. Though we shall not prove it, the element g' asserted to exist by the definition is unique for each $0 < g \in G$, and in fact $0 \leq g' \leq g$ in all cases. Also, not every type $\mathcal{W}(P)$ set X is type \mathcal{Z} , nor is every type \mathcal{Z} set Y necessarily type $\mathcal{W}(P)$. For example, take $G = C[-1, 1]$, the l -group of continuous real-valued functions on the interval $[-1, 1]$, let $P = \{g \in G \mid \text{supp}(g) \subseteq (0, 1)\}$, let $X = \{g \in P^+ \mid (r)g \leq 1/r, r \in (0, 1)\}$, and let $Y = \{g \in P^+ \mid (r)g \leq r/(1 - r), r \in (0, 1)\}$. Nevertheless, the notions of type \mathcal{Z} set and type $\mathcal{W}(P)$ set coincide for summands P .

PROPOSITION 6.9. *Suppose P is a summand of the l -group G . A set $X \subseteq P^+$ is of type $\mathcal{W}(P)$ if and only if X is type \mathcal{Z} in P .*

PROOF. Suppose X is of type $\mathcal{W}(P)$. Then X is type \mathcal{U} in G and therefore also in P . Consider $0 < p \in P$, and let p' be the element provided by the definition of X being type $\mathcal{W}(P)$. Then $p' \in P$ because $p - p' \in P$, $X \wedge p \leq p'$, and

$$\bigvee\{(p' - x \wedge p)^\perp : x \in X\} = 1_{\mathcal{Z}},$$

where \mathcal{Q} is the polar algebra of P . By Lemma 6.1 $\vee(X \wedge p) = p' \in P$, meaning that X is type \mathcal{Z} in P .

Now suppose X is type \mathcal{Z} in P . Then $\vee\{((X - x) \vee 0)^\perp : x \in X\} = 1_{\mathcal{Q}}$ and $X \subseteq P$ imply $\vee((X - x) \vee 0)^\perp = 1_{\mathcal{P}}$, where \mathcal{P} is the polar algebra of G , meaning X is type \mathcal{Y} in G . Given $0 \leq g \in G$ let $g = y + z$, where $y \in P$ and $z \in P^\perp$, and let $g' = w + z$, where $w = \vee(X \wedge y)$ exists in P because X is type \mathcal{Z} in P . Then $g - g' = y - w \in P$, and $X \wedge g = X \wedge (y + z) \leq X \wedge y + z \leq w + z = g'$. To show $P \subseteq \vee\{(g' - x \wedge g)^\perp : x \in X\}$ in \mathcal{P} first observe that $(x_1 \wedge y) - x \wedge y \leq (x_1 - x)^\perp$ for $x_1, x \in X$, from which follows $\vee\{((X \wedge y - x \wedge y) \vee 0)^\perp : x \in X\} = 1_{\mathcal{P}}$. A similar argument shows $\vee((-x \wedge y + X \wedge y) \vee 0)^\perp = 1_{\mathcal{P}}$, and so $X \wedge y$ is type \mathcal{Y} . By Lemma 6.1 $\vee(w - x \wedge y)^\perp = 1_{\mathcal{P}}$, from which follows $\vee(g' - x \wedge g)^\perp \supseteq P$. \square

COROLLARY 6.10. $X \subseteq G^+$ is of type $\mathcal{W}(G)$ if and only if X is type \mathcal{Z} .

COROLLARY 6.11. If G is sup \mathcal{W} complete, then every summand is sup \mathcal{Z} complete. If G is projectable and every polar is sup \mathcal{Z} complete, then G is sup \mathcal{W} complete.

LEMMA 6.12. Suppose P is a polar of the l -group G , and that $X, Y \subseteq P^+$ are sets of type $\mathcal{W}(P)$. Then $X + Y$ is of type $\mathcal{W}(P)$.

PROOF. First we must show $X + Y$ to be type \mathcal{Y} . We know that $\vee X = x_0 \in G_{\mathcal{Q}}$ and $\vee Y = y_0 \in G_{\mathcal{Q}}$, hence by Lemma 6.1 and its analogue $\vee\{(y_0 - y)^\perp : y \in Y\} = \vee\{(-x + x_0)^\perp : x \in X\} = 1_{\mathcal{P}}$. It follows that

$$\begin{aligned} 1_{\mathcal{P}} &= \vee\{((y_0 - y)^\perp \cap (-x + x_0)^\perp) : x \in X, y \in Y\} \\ &= \vee((y_0 - y) \vee (-x + x_0))^\perp = \vee((y_0 - y) + (-x + x_0))^\perp. \end{aligned}$$

Upon conjugation by x_0 we obtain $\vee((x_0 + y_0) - (x + y))^\perp = 1_{\mathcal{P}}$, which proves $X + Y$ to be type \mathcal{Y} in $G_{\mathcal{Q}}$ by Lemma 6.1. By Proposition 5.16 of [1], $X + Y$ is also type \mathcal{Y} in the order dense l -subgroup G .

Now consider $0 \leq g \in G$ and let g_x and g_y be the elements of G provided by the type $\mathcal{W}(P)$ property of X and Y , respectively. Let $g' = (g_x + g_y) \wedge g$. Then $0 \leq g - g' = (g - g_y - g_x)^\perp \leq (g - g_y)^\perp \in P$, so $g - g' \in P$. And for $x \in X$ and $y \in Y$, $(x + y) \wedge g \leq (x \wedge g) + (y \wedge g) \leq g_x + g_y$, which implies $(X + Y) \wedge g \leq g'$. Finally, to show $P \subseteq \vee\{(g' - (x + y) \wedge g)^\perp : x \in X, y \in Y\}$ consider $0_{\mathcal{P}} \neq T_1 \subseteq P$. Choose $x \in X$ such that $0_{\mathcal{P}} \neq (g_x - x \wedge g)^\perp \cap T_1 = T_2$. The fact that $x \in P$ implies $0_{\mathcal{P}} \neq -x + T_2 + x \subseteq P$, hence there must be some $y \in Y$ such that $0_{\mathcal{P}} \neq (g_y - y \wedge g)^\perp \cap (-x + T_2 + x) = T_3$. Note that $0_{\mathcal{P}} \neq x + T_3 - x \subseteq T_2$. We claim $x + T_3 - x \subseteq (g' - (x + y) \wedge g)^\perp$. For if not then there is some $r \in x + T_3 - x$ such that $0 < r \leq g' - (x + y) \wedge g$. Let Q be a value of r , and observe that $Q + (x + y) \wedge g < Q + (g_x + g_y) \wedge g$. Since $r \in (g_x - x \wedge g)^\perp \setminus Q$, $Q + g_x = Q + x \wedge y$. If $Q + g \leq Q + x$, then $Q + g_x = Q + g$ and $Q + (x + y) \wedge g = Q + g = Q + (g_x + g_y) \wedge g$, a contradiction. Therefore $Q + g_x = Q + x < Q + g$. Now $-x + r + x \in (g_y - y \wedge g)^\perp \setminus R$ implies $R + g_y = R + y \wedge g$, where $R = -x + Q + x$. If $R + y \leq R + g$ we get $R + g_y = R + y$, which gives the contradiction

$Q + g_x + g_y = Q + x + y$. If $R + y > R + g$, then $R + g_y = R + g$, so $Q + x + y > Q + x + g \geq Q + g$ and $Q + g_x + g_y = Q + x + g_y = Q + x + g \geq Q + g$, and we again have the contradiction $Q + (x + y) \wedge g = Q + g = Q + (g_x + g_y) \wedge g$. \square

Theorem 6.14 is important and explains the significance of type $\mathcal{W}(P)$ subsets for our purposes. For its proof it is helpful to recast the definition of such sets slightly in Lemma 6.13, which in turn requires some terminology. Given an element g and polar P of G we call h the projection of g on P if $h \in P$ and $g - h \in P^\perp$. h need not exist, but when it does we write h as $g\nu P$. This terminology extends the use of ν in previous sections in the sense that *bva* there could also be written $b\nu[a]$ here.

LEMMA 6.13. Suppose $X \subseteq G^+$ and $P \in \mathcal{P}$. Then X is of type $\mathcal{W}(P)$ if and only if there is an element $t \in G_{\mathcal{W}}$ satisfying the following conditions.

- (a) $X \leq t \in P^{\perp\perp}$.
- (b) $\bigvee\{(t - x)^\perp : x \in X\} = 1_{\mathcal{P}}$.
- (c) $(g - t)^+\nu P \in G$ for all $g \in G^+$.

PROOF. Suppose X is of type $\mathcal{W}(P)$; since X is type \mathcal{U} there is some element $t \in G_{\mathcal{W}}$ such that $\bigvee X = t$, and $t \in P^{\perp\perp}$ because $P^{\perp\perp}$ is a closed convex l -subgroup of $G_{\mathcal{W}}$. By Lemma 6.1, $\bigvee(t - x)^\perp = 1_{\mathcal{P}}$. Given $g \in G^+$ find $g' \in G$ such that $g - g' \in P$ and $P \subseteq \bigvee(g' - x \wedge y)^\perp$. We claim $g - g' = (g - t)^+\nu P$, for which we must prove

$$\begin{aligned} (g - t)^+ - (g - g') &= (g - t) \vee 0 + g' - g \\ &= (g' - t) \vee (g' - g) = g' - t \wedge g \in P^\perp. \end{aligned}$$

If not then $P \cap (g' - t \wedge g)^\perp = R_1 \neq 0_{\mathcal{P}}$, and so $(g' - x_1 \wedge g)^\perp \cap R_1 = R_2 \neq 0_{\mathcal{P}}$ for some $x_1 \in X$, hence $(t - x_2)^\perp \cap R_2 \neq 0_{\mathcal{P}}$ for some $x_2 \in X$. Note that

$$t \wedge g - x_2 \wedge g = ((t - x_2) \wedge (g - x_2)) \vee ((t - g) \wedge 0) \leq t - x_2,$$

so $(t \wedge g - x_2 \wedge g)^\perp \cap R_2 \neq 0$. By replacing x_1 and x_2 by $x = x_1 \vee x_2$ we get $(g' - x \wedge g)^\perp \cap (t \wedge g - x \wedge g)^\perp \cap (g' - t \wedge g)^\perp \neq 0_{\mathcal{P}}$. But this is a contradiction since

$$\begin{aligned} (g' - x \wedge g)^\perp \cap (t \wedge g - x \wedge g)^\perp &\subseteq (g' - x \wedge g - (t \wedge g - x \wedge g))^\perp \\ &= (g' - t \wedge g)^\perp. \end{aligned}$$

Conversely suppose X and t satisfy conditions (a), (b), and (c). Then X is type \mathcal{U} , $\bigvee X = t$ by Lemma 6.1, and $X \subseteq P^{\perp\perp} \cap G = P$. Given $g \in G^+$ let $g' = -(g - t)^+\nu P + g \in G$. Then $g - g' = (g - t)^+\nu P \in P$. To show $X \wedge g \leq g'$ consider an $x \in X$ and arbitrary prime Q such that $Q < Q + x \wedge g$. Since $P \not\subseteq Q$ it follows that $(g - t)^+\nu P^\perp \in P^\perp \subseteq Q$, hence

$$\begin{aligned} Q + g' &= Q - (g - t)^+\nu P + g = Q - (g - t)^+ + g \\ &= Q = t \wedge g \geq Q + x + g. \end{aligned}$$

To show $P \subseteq \bigvee\{(g' - x \wedge g)^\perp : x \in X\}$ consider $0_{\mathcal{P}} \neq R \subseteq P$ and find $x \in X$ such that $(t - x)^\perp \cap R \neq 0_{\mathcal{P}}$. We claim $(t - x)^\perp \cap P \subseteq (g' - x \wedge g)^\perp \cap P$. For if not there is some $0 < p \in (t - x)^\perp \cap P$ such that $p \leq g' - x \wedge g$. Let Q be a

value of p . Again $P \not\subseteq Q$ implies $(g - t)^+ \nu P^\perp \in P^\perp \subseteq Q$, so

$$\begin{aligned} Q + x \wedge g &< Q + g' = Q - (g - t)^+ \nu P + g \\ &= Q - (g - t)^+ + g = Q + t \wedge g. \end{aligned}$$

But $p \wedge (t - x) = 0$ implies $t - x \in Q$ or $Q + t = Q + x$, hence $Q + x \wedge g = Q + t \wedge g$, a contradiction. \square

If $X \subseteq G^+$ is of type $\mathcal{W}(P)$ and $\forall X = t \in G_{\mathcal{W}}$, then G need not be convex in the l -subgroup $\langle G \cup \{t\} \rangle \leq G_{\mathcal{W}}$. The following theorem asserts, however, that t must be the projection on P of an element h in an extension H in which G is a convex l -subgroup. In the statement of this theorem $\bigvee\{(h - x)^\perp : x \in X\}$ refers to the supremum in the Boolean algebra \mathcal{P} of polars of H .

THEOREM 6.14. *Suppose G is an abelian l -group with polar P and subset $X \subseteq G^+$. Then X is of type $\mathcal{W}(P)$ if and only if there is an l -group H and element $0 < h \in H$ such that G is a convex l -subgroup of H , $h \geq X \subseteq P$, and*

$$(G - H)^+ \subseteq P \subseteq \bigvee\{(h - x)^\perp : x \in X\}.$$

PROOF. Suppose G is a convex l -subgroup of an l -group H as above. To prove that X is of type \mathcal{W} consider $T \neq 0_{\mathcal{P}}$; we seek $x \in X$ such that $((X - x) \vee 0)^\perp \cap T \neq 0_{\mathcal{P}}$. If $T \not\subseteq P$ we are done, for $((X - x) \vee 0)^\perp \supseteq P^\perp$ for every $x \in X$. Assuming $T \subseteq P$, find $x \in X$ such that $T \cap (h - x)^\perp \neq 0_{\mathcal{P}}$. Since $(x_1 - x) \vee 0 \leq h - x$ for all $x_1 \in X$, it follows that $((X - x) \vee 0)^\perp \cap T \neq 0_{\mathcal{P}}$. To prove that X is of type $\mathcal{W}(P)$ consider an arbitrary $0 \leq g \in G$ and let $g' = h \wedge g$, an element of G by virtue of the convexity of G in H . Then $g - g' = (g - h)^+ \in P$ by hypothesis, and $X \wedge g \leq h \wedge g = g'$. To demonstrate that $P \subseteq \bigvee\{(g' - x \wedge g)^\perp : x \in X\}$ consider $0_{\mathcal{P}} \neq T \subseteq P$ and find $x \in X$ such that $(h - x)^\perp \cap T \neq 0_{\mathcal{P}}$. Since

$$0 \leq h \wedge g - x \wedge g = ((h - g) \wedge (g - x)) \vee ((h - g) \wedge 0) \leq h - x,$$

it follows that $(g' - x \wedge g)^\perp \cap T \neq 0_{\mathcal{P}}$.

Suppose X is of type $\mathcal{W}(P)$. Let K be $\mathbf{Z} \times \bar{G}$, the splitting extension of the orthocompletion \bar{G} of G by the integers \mathbf{Z} determined by defining $\sigma(n)$ to be the identity l -automorphism on \bar{G} and $\pi(n) = [n] = P^\perp \in \mathcal{S}(\bar{G})$ for all $0 \neq n \in \mathbf{Z}$. Let $\forall X = t \in \bar{G}$ and let H be the subgroup $\{(n, nt + g) : n \in \mathbf{Z}, g \in G\}$. We claim that H is an l -subgroup of K , the proof of which requires showing $(n, nt + g) \vee (0, 0) \in H$ for all $n \in \mathbf{Z}$ and $g \in G$. Consider first the case in which $n > 0$. By Lemma 3.9

$$\begin{aligned} (n, nt + g) \vee (0, 0) &= (n, (nt + g)\nu n + (nt + g)^+ \mu n) \\ &= (n, nt + g - (nt + g)^- \mu n) \\ &= (n, nt + g + (-g - nt)^+ \mu n) \\ &= (n, nt + g + ((-g)^+ - nt)^+ \nu P). \end{aligned}$$

The point is that by Lemma 6.12 $nX = \{\sum_{i=1}^n x_i : x_i \in X\}$ is type $\mathcal{W}(P)$ with supremum nt , and $((-g)^+ - nt)^+ \nu P$ exists in G by Lemma 6.13. Therefore $(n, nt + g) \vee (0, 0) \in H$ in this case. In case $n = 0$ we have

$$(0, g) \vee (0, 0) = (0, g \vee 0) \in H,$$

and in case $n < 0$ we have

$$\begin{aligned} (n, nt + g) \vee (0, 0) &= (0, (nt + g)^+ \mu n) = (0, (nt + g)^+ \nu P) \\ &= (0, (g^+ - (-n)t)^+ \nu P). \end{aligned}$$

As before, $kX = \{\sum_{i=1}^k x_i : x_i \in X\}$ is type $\mathcal{W}(P)$ with supremum kt , where $k = -n$, and so $(g^+ - kt)\nu P \in G$ by Lemma 6.13. This completes the proof that H is an l -subgroup of K , and G is clearly convex in H .

Let $h = (1, t) \in H$; we must verify that $(G - h)^+ \subseteq P \subseteq \bigvee \{(h - x)^\perp : x \in X\}$. Now for any $g \in G$,

$$(g - h)^+ = (-1, g - t) \vee (0, 0) = (0, (g - t)^+ \nu P) = (0, (g^+ - t)^+ \nu P) \in P.$$

To show $P \subseteq \bigvee (h - x)^\perp$ consider $0_\emptyset \neq R \subseteq P$, then find $x \in X$ such that $(t - x)^\perp \cap R \neq 0_\emptyset$. For $0 < p \in R \cap (t - x)^\perp$ we have

$$p \wedge (h - x) = (0, p) \wedge (1, t - x) = (0, (t - x) \wedge p) = (0, 0).$$

This completes the proof. \square

PROPOSITION 6.15. *An abelian l -group G which splits off from below in every abelian l -group in which it is a convex l -subgroup must be sup \mathcal{W} complete. In this case any polar P which admits a type $\mathcal{W}(P)$ subset is a cardinal summand.*

PROOF. Suppose P is a polar and X a type $\mathcal{W}(P)$ subset of G and construct H with element h as in the proof of Theorem 6.14. Suppose for contradiction that $\bigvee X = t \in \overline{G} \setminus G$. We claim $\{0\}$ is maximal above G . Now an arbitrary element of $H^+ \setminus G$ has the form $nh + g = (n, nt + g)$ for $0 < n \in \mathbb{Z}$ and $g \in G$. Note that $(nt + g)\nu P = nt + g\nu P \neq 0$ in \overline{G} , for otherwise $0 < t \leq nt \leq |g|$ implies $t \in G$. By Proposition 3.15 $(n, nt + g)$ has a value in $\Gamma(G)$, hence G does not lie below $\langle (n, nt + g) \rangle$. This proves the claim, and since the claim directly violates the hypothesis of the proposition, we can only conclude that $t \in G$, or that G is sup \mathcal{W} complete.

Observe that $h - t = (1, 0)$ lies above G , and that $H(h - t) \cap G = P^\perp$, so $P \in \mathcal{S}(G)$ by Proposition 1.1. \square

We have arrived at last at the main theorem.

THEOREM 6.16. *An abelian l -group G splits off from below in every abelian l -group in which it is a convex l -subgroup if and only if G is divisible, sup \mathcal{W} complete, and has the SSP.*

PROOF. Suppose G has the three listed properties and is a convex l -subgroup of the abelian l -group H . Let U be an l -subgroup of H maximal above G , and consider an arbitrary $0 < h \in H$. We shall produce an element $g \in G$ such that $h - g \in U$. All polar computations will be done in \mathcal{P} , the Boolean algebra of polars of H .

For each $u \in U$ and positive integer n let

$$\begin{aligned} P(n, u) &= P = \bigvee \{((g - |nh - u|) \vee 0)^{\perp+} : g \in G^+\}, \\ X_1 &= P^+ \wedge (nh - u)^+, \quad \text{and} \quad X_2 = P^+ \wedge (u - nh)^+. \end{aligned}$$

We claim $P \subseteq \bigvee \{((nh - u)^+ - x_1)^\perp : x_1 \in X_1\}$. To verify this consider $0_\varnothing \neq T \subseteq P$ and find $0 < g \in G$ such that $((g - |nh - u|) \vee 0)^\perp \cap T \neq 0_\varnothing$, then find $0 < t \in T$ with $t \leq (g - |nh - u|)^+ \leq (g - (nh - u)^+)^+$. Let $x_1 = g \wedge (nh - u)^+ \in X_1$. Then

$$t \wedge ((nh - u)^+ - x_1) = t \wedge ((nh - u)^+ - g)^+ = 0,$$

proving the claim. By Theorem 6.14 X_1 , and analogously X_2 , are of type $\mathcal{W}(P)$, so by Proposition 6.15 there are elements $a, b \in G$ with $\bigvee X_1 = a$ and $\bigvee X_2 = b$, and $P(n, u)$ is a summand. By the divisibility of G there is some $g(n, u) \in G$ such that $ng(n, u) = a - b$.

We claim $n(h - g(n, u)) - u$ lies above G . To see this, first observe that $(nh - u)^+ - a \in P^\perp$ and $-(nh - u)^- - b \in P^\perp$, hence $n(h - g(n, u)) - u \in P^\perp$. If there were some prime Q and element $g \in G$ such that $Q < Q + |n(h - g(n, u)) - u| < Q + g$, then

$$Q + |nh - u| - |ng(n, u)| \leq Q + |n(h - g(n, u)) - u| < Q + g,$$

so $Q + |nh - u| < Q + g + |ng(n, u)|$, meaning $((g + |ng(n, u)|) - |nh - u|) \vee 0$ is an element of $P \setminus Q$. Therefore $nh - u - ng(n, u) \in P^\perp \subseteq Q$, contradicting $Q < Q + |n(h - g(n, u)) - u|$.

We claim the map θ which takes each $P(n, u)$ to $g(n, u)$ is consistent. That is, $g(n, u) - g(m, v) \in [P(n, u) \cap P(m, v)]^\perp$ for all $u, v \in U$ and positive integers m and n . For if not there is some prime Q of G such that $Q + g(n, u) \neq Q + g(m, v)$ and $Q \not\supseteq P(n, u) \cap P(m, v)$. We have seen in the previous paragraph that $n(h - g(n, u)) - u$ and $m(h - g(m, v)) - v$ lie in P^\perp . Therefore $\overline{Q} + nh - u = \overline{Q} + ng(n, u)$ and $\overline{Q} + mh - v = \overline{Q} + mg(m, v)$, where \overline{Q} is the unique prime subgroup of H whose intersection with G is Q . Thus

$$\begin{aligned} \overline{Q} + mn(g(m, v) - g(n, u)) &= \overline{Q} + ((mnh - nv) - (mnh - mu)) \\ &= \overline{Q} + (mu - nv). \end{aligned}$$

It cannot be that $mu - nv \in U$ fails to lie in \overline{Q} , since in this case the value of $mu - nv$ containing \overline{Q} would correspond to the value of $mn(g(m, v) - g(n, u))$ containing Q , and $mu - nv$ would have a value in $\Gamma(G)$ contrary to the assumption that U lies above G . But then $\overline{Q} + g(n, u) = \overline{Q} + g(m, v)$, contrary to the assumption that $Q + g(n, u) \neq Q + g(m, v)$. This completes the proof that θ is consistent.

Let $g \in G$ result from applying the SSP to θ ; explicitly, $g - g(n, u) \in P(n, u)^\perp$ for all u and all n . If $h - g \notin U$, then by Proposition 1.9 there is some $u \in U$ and positive integer n such that $n(h - g) - u$ has value Q in $\Gamma(G)$. Let $0 < k \in G$ satisfy $Q + k > Q + |n(h - g) - u|$. Since $Q + (k + ng) > Q + (nh + u)$ and $Q + (k - ng) > Q + (u - nh)$ it follows that $Q + (k + |ng|) > Q + |nh - u|$, that is $(k + |ng| - |nh - u|)^+ \in P(n, u) \setminus Q$. Since $Q \not\supseteq P(n, u)$, $g - g(n, u) \in P(n, u)^\perp \subseteq R$ for all prime subgroups $R \subseteq Q$. Therefore

$$R + (n(h - g) - u) = R + (n(h - g(n, u)) - u)$$

for all prime subgroups $R \subseteq Q$, and in particular, $n(h - g(n, u)) - u$ has value $Q \in \Gamma(G)$. This contradicts the fact that $n(h - g(n, u)) - u$ lies above G . \square

COROLLARY 6.17. *A divisible abelian l -group which is $\sup \mathcal{Y}$ complete splits off from below in every abelian l -group in which it is a convex l -subgroup.*

THEOREM 6.18. *Suppose G is a divisible abelian $\sup \mathcal{Y}$ complete l -group. Then a convex l -subgroup L splits off from below in G if and only if L is maximal below if and only if L is closed.*

PROOF. Given a closed convex l -subgroup L , we observe that each type \mathcal{Y} subset $X \subseteq L^+$ remains type \mathcal{Y} in G . Therefore L is $\sup \mathcal{Y}$ complete and divisible. \square

COROLLARY 6.19. *The maximal below l -subgroups of a divisible abelian $\sup \mathcal{Y}$ complete l -group form a lattice under the inclusion order.*

COROLLARY 6.20. *Suppose G is a divisible abelian $\sup \mathcal{Y}$ complete l -group, and that U_1 and U_2 are both maximal above the convex l -subgroup L . Then there is an l -automorphism θ of G such that θ is the identity on the closure of L and $U_1\theta = U_2$.*

LEMMA 6.21. *Suppose G is an abelian l -group. If G is divisible, then so is $G_{\mathcal{Y}}$.*

PROOF. Given X a type \mathcal{Y} subset of G and positive integer n let $Y = \{z \in G \mid nz \in X\}$. Then Y must be type \mathcal{Y} , for if $x_1, x_2 \in X$ and if $ny_i = x_i$, then

$$((x_1 - x_2) \vee 0)^+ = ((ny_1 - ny_2) \vee 0)^+ = (n(y_1 - y_2) \vee 0)^+ = ((y_1 - y) \vee 0)^+,$$

meaning $((X - x_i) \vee 0)^+ = ((Y - y_i) \vee 0)^+$. To prove the lemma form $G_{\mathcal{Y}}$ inductively as follows. Let $G_0 = G$, and if $G_\alpha = G_{\mathcal{Y}}$ let $G_{\alpha+1} = G_\alpha$. If $G_\alpha \neq G_{\mathcal{Y}}$ choose $0 < x \in G_{\mathcal{Y}} \setminus G_\alpha$ and let $G_{\alpha+1}$ be the l -subgroup of $G_{\mathcal{Y}}$ generated by $G_\alpha \cup \{qx \mid q \text{ rational}\}$. Let $G_\beta = \bigcup_{\alpha < \beta} G_\alpha$ for limit ordinals β . A simple induction shows G_α divisible for all α , hence $G_{\mathcal{Y}}$, which is G_α for sufficiently large ordinals α , is also divisible. \square

In the next proposition G^d designates the l -group divisible hull of G .

PROPOSITION 6.22. *Any abelian l -group G is a large l -subgroup of some abelian l -group H which splits off from below in every abelian l -group in which it (H) is a convex l -subgroup. H may be taken to be $(G^d)_{\mathcal{Y}}$.*

We close this section by giving an example of an l -group G which is $\sup \mathcal{W}$ complete but lacks the SSP. Though G is simple in structure in the sense that $\sum_{n=1}^{\infty} \mathbb{Z}_n \leq G \leq \prod_{n=1}^{\infty} \mathbb{Z}_n$, the construction itself uses the Continuum Hypothesis: let $\{f_\alpha \mid \alpha < \aleph_1\}$ be an enumeration of all functions from N into N , where $N = \{1, 2, 3, \dots\}$. Define $G_0 = \sum_{n=1}^{\infty} \mathbb{Z}_n$. Given countable G_α define $G_{\alpha+1}$ as follows. Let $\{w_k(x) \mid k \in N\}$ be an enumeration of all l -group terms of the form $\bigvee_{i=1}^n \bigwedge_{j=1}^m (g_{ij} + n_{ij}x)$, where $g_{ij} \in G_\alpha$, $n_{ij} \in \mathbb{Z}$, and x is an indeterminate. Iteratively define a function $f: N \rightarrow N$ such that $(n)f \geq (n)f_\alpha$ for all $n \in N$, and such that $(n)(w_k(f)) \neq 1$ for all $1 \leq k \leq n \in N$. Let $G_{\alpha+1}$ be the l -subgroup of $\prod_{n=1}^{\infty} \mathbb{Z}_n$ generated by $G_\alpha \cup \{f\}$. Define $G_\gamma = \bigcup \{G_\alpha \mid \alpha < \gamma\}$ for limit ordinals $\gamma \leq \aleph_1$. Let G be the projectable hull of G_{\aleph_1} . That is, G consists of those elements $g \in \prod \mathbb{Z}_n$ for which there exist disjoint sets $X_1, X_2, \dots, X_n \subseteq N$ such that $\bigcup_{i=1}^n X_i = N$, and such that for each X_i there is some $g_i \in G_{\aleph_1}$ with $(n)g = (n)g_i$ for all $n \in X_i$. Observe

that for any $g \in G$ $\{n \in N: (n)g = 1\}$ is finite since this condition holds for all $g \in G_{\aleph_1}$. Since $\Sigma \mathbf{Z}_n \leq G \leq \prod \mathbf{Z}_n$ and since every polar of G is a cardinal summand, G cannot possess the SSP.

To demonstrate that G is $\sup \mathcal{W}$ complete we need only show that every polar is $\sup \mathcal{Z}$ complete. Consider a type \mathcal{Z} subset X of polar P . Because X is type \mathcal{U} , $\{(n)p: p \in X\}$ is a bounded set for each index $n \in N$. Let $\alpha < \aleph_1$ be such that $(n)f_\alpha = V\{(n)p: p \in X\}$ for each $n \in N$. There is by construction some element $g \in G$ with $g \geq f_\alpha$, and by projecting it on P we may assume $g \in P$. By definition of type \mathcal{Z} , $\forall X = V(X \wedge g)$ exists in P . We have shown P to be $\sup \mathcal{Z}$ complete, and hence G to be $\sup \mathcal{W}$ complete.

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